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THE THEORY OF
CONTINUOUS STRUCTURES AND ARCHES

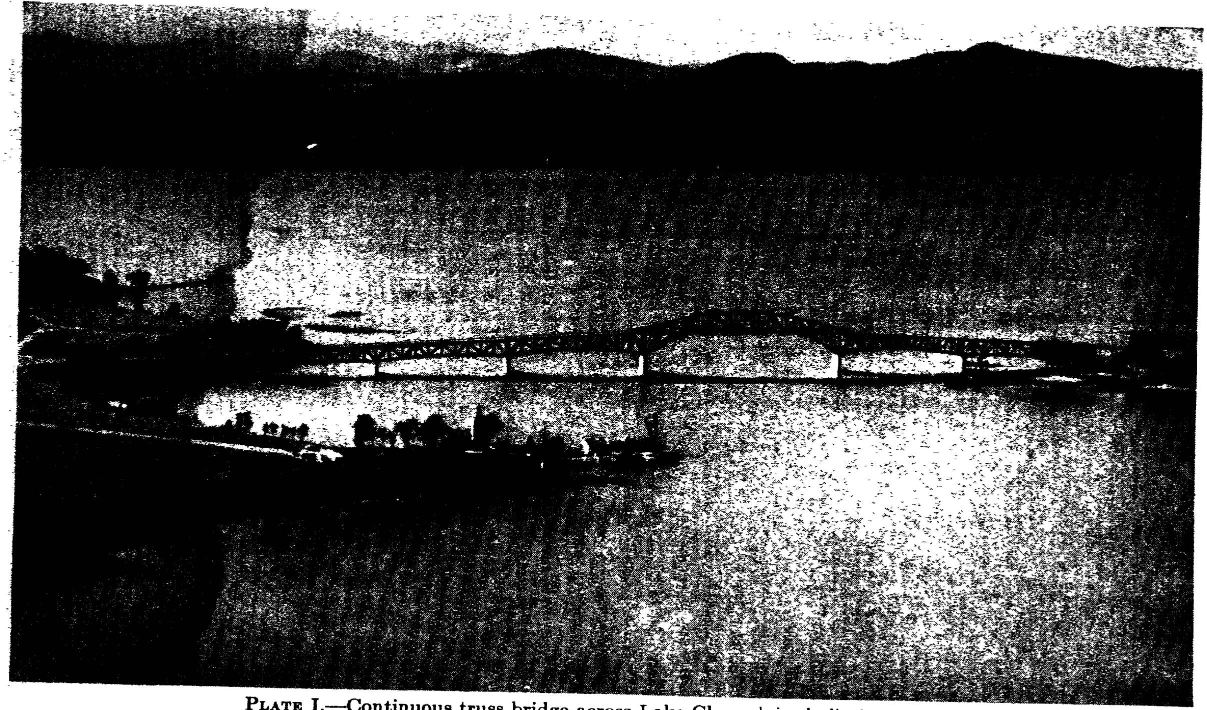

Plate I.—Continuous truss bridge across Lake Champlain, built 1928–1929.

(Frontispiece.)

THE THEORY OF CONTINUOUS STRUCTURES AND ARCHES

BY

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PREFACE

The subject matter of this book is based upon the author's experience in designing bridges of the types covered by the text and in teaching advanced courses in structural engineering to graduate students at the Massachusetts Institute of Technology.

The theories are exemplified by numerical examples arranged to conform to the needs of a designing engineer with a staff competent to carry on routine calculations. The illustrations, unless otherwise indicated, are of bridges designed and supervised during construction by the author and his associates.

Special credit is given in the text to certain papers, books and individuals, and additional acknowledgment is hereby made for helpful suggestions by various members of the instructing staff of the Massachusetts Institute of Technology, by graduate students, and by partners and members of the engineering staff of Fay, Spofford and Thorndike. Particular credit is due Charles H. Norris, Jr., S. M., for valuable assistance in preparing the manuscript for publication.

CHARLES M. SPOFFORD.

BOSTON, MASS.,
January, 1937.

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THE THEORY OF CONTINUOUS STRUCTURES AND ARCHES

CHAPTER I

CONTINUOUS STRUCTURES

1. General.—A continuous structure is one that is supported at more than two points and is, in general, statically indeterminate with respect to the outer forces. Such structures are used widely in reinforced concrete where continuity is difficult to avoid and are being used more and more frequently for plate-girder viaducts and bridges.

Continuous-truss bridges have not been built in the United States so extensively as have simple span and cantilever bridges, probably because of the inadequate knowledge of the theory of such structures among engineers of the older generation and the fact that differential pier settlements cause changes in the live stresses in the ordinary statically indeterminate type of continuous bridge. The theory of continuous trusses is now well understood, and its application presents no serious difficulty. Continuous bridges of the statically indeterminate type, however, should not be built on foundations where serious differential pier settlements are likely to occur, although slight settlements have little effect upon the stresses, as shown in the computations given in Art. 13 *et seq.* Continuous trusses that are statically determinate are not open to this criticism; such trusses are discussed in Art. 20 *et seq.*

A continuous bridge of the statically indeterminate type usually has advantages in rigidity and economy[1] and in the fewer number of expansion joints as compared with end-supported or cantilever bridges. Its use also sometimes permits a decrease in the width of intermediate piers and a reduction in the number of end shoes, end floor beams, etc. Continuous bridges have all the advantages of cantilever bridges for locations over navigable waters where clearance must be maintained during the construction period.

---

[1] See also Art. 22.

Numerous continuous bridges have been built in recent years, among them the following:

Queensboro Bridge (Blackwell's Island), 1909.
  Reference: *Bridge Eng.*, Vol. I, p. 595.
    Iron Age, June 17, 1909.
    Eng. News-Rec., June 17, 1909.
  Spans: 470, 1182, 630, 984, 459 ft.
Sciotoville Bridge over Ohio River, 1917.
  Reference: *Eng. News-Rec.*, May 17, 1917, p. 343.
  Spans: 775, 775 ft.
Bessemer and Lake Erie Railroad Bridge over Allegheny River, 1918.
  Reference: *Eng. News-Rec.*, May 2, 1918, p. 848.
  Spans: 272, 520, 347, 347, 350, 272 ft.
Hudson Bay Railroad Bridge over Nelson River, 1918.
  Reference: *Eng. News-Rec.*, Aug. 29, 1918, p. 388.
  Spans: 300, 400, 300 ft.
Cincinnati, New Orleans and Texas Pacific Railroad Bridge over Ohio River, 1922.
  Reference: *Trans. Am. Soc. C.E.*, Vol. 85, 1922, p. 964.
  Spans: 300, 300, 516 ft.
Lake Champlain Bridge, 1931.
  Reference: *Trans. Am. Soc. C.E.*, Vol. 98, 1933, p. 622.
  Spans: 290, 434, 290 ft.
Little Bay Bridge, 1934.
  Reference: *J. Boston Soc. C.E.*, Vol. 22, No. 1, January, 1935.
  Spans: 200, 275, 200 ft.
Cape Cod Highway Bridges at Bourne, Mass., 1935.
  Reference: *Military Eng.*, Vol. 28, No. 159, May–June, 1936.
  Spans: 396, 616, 396 ft.
Clinch River Bridge near Knoxville, Tenn., 1935.
  Spans: 253 ft. 11¾ in. and 253 ft. 1¼ in.

2. Continuous Beams and Girders. Method of Design.—The reactions on continuous girders can be accurately determined by the "theorem of three moments," provided the moment of inertia and the modulus of elasticity of the material are each constant throughout the entire length of the girder, a condition that sometimes exists. If the moments of inertia and the moduli of elasticity are different in the separate spans or vary in one or more spans, the reactions cannot be accurately determined until the cross-section areas are known; hence an accurate determination by this method of the stresses in such structures can be accomplished only by a series of approximations, the reactions first being approximately determined, the stresses and areas computed, and the computations revised to correspond to the new areas, the process being repeated as often as is necessary to obtain sufficiently accurate results.

3. Derivation of the "Three-moment Equation."—The three-moment equation derived herein is applicable only to cases where

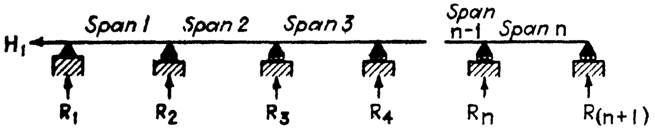


FIG. 1.

the product of the cross section and modulus of elasticity in each span of the beam is constant. The derivation of the three-moment equation for other cases will not be given in this book, and the reader is referred to the references at the end of the chapter for further information on this subject.

Let the beam shown in Fig. 1 have n spans. There will then be $(n + 2)$ unknown reactions, all the supports but one being

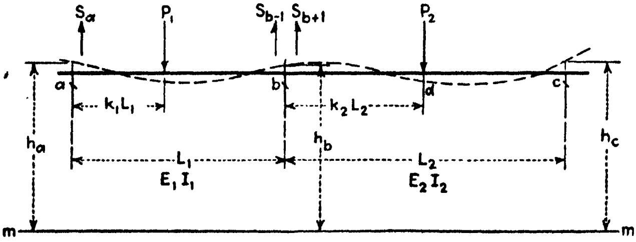


FIG. 2.

on rollers; hence $(n - 1)$ equations must be obtained from conditions other than those of statics. These equations may be deduced from the differential equation of the elastic curve $d^2y/dx^2 = M/EI$, by the method that follows, each of the $(n - 1)$ resulting equations involving the moments at three adjoining supports.

Let Fig. 2 represent a portion of a continuous beam, the entire structure having n spans, and the section under consideration including any portion of the girder supported upon three adjoining supports. The axis of the unstressed beam is assumed to be straight, and the supports level.

Now assume the beam to be loaded with the concentrated loads P_1 and P_2, and let the axis of the stressed beam be represented by the dotted line in Fig. 2.

Let M_a, M_b, and M_c be the moments upon the beam at the supports a, b, and c, and assume these as positive when causing compression in the upper fibre of the portion of the beam under consideration.

Let S_a, S_{b-1}, and S_{b+1} be the shears at infinitesimal distances from the supports a and b, and let these be assumed as positive when acting as shown.

Let M equal the moment at any section of the beam.

Let t_c, t_{b+1}, and t_{b-1} equal the tangents at c and at infinitesimal distances on either side of b of the angle between the neutral axis of the stressed beam and its original position.

Let h_a, h_b, h_c equal normal distances of points a, b, and c on the axis of the stressed beam above some assumed axis mm' parallel to the axis of the unstressed beam and below it.

For the portion of the beam between b and c, the moment at a distance x from b is given by the following equations:

$$M = E_2I_2\frac{d^2y}{dx^2} = M_b + S_{b+1}x \text{ for portion } b \text{ to } d \tag{1}$$

$$M = E_2I_2\frac{d^2y}{dx^2} = M_b + S_{b+1}x - P_2(x - k_2L_2) \text{ for portion } d \text{ to } c \tag{2}$$

From Eq. (1) by integration we obtain

$$E_2I_2\frac{dy}{dx} = M_bx + S_{b+1}\frac{x^2}{2} + C_1E_2I_2 \tag{3}$$

and

$$E_2I_2y = M_b\frac{x^2}{2} + S_{b+1}\frac{x^3}{6} + C_1E_2I_2x + C_2E_2I_2 \tag{4}$$

When $x = 0$,

$$\frac{dy}{dx} = t_{b+1} \quad \text{and} \quad y = h_b$$

Therefore

$$C_1 = t_{b+1} \quad \text{and} \quad C_2 = h_b$$

Hence

$$E_2I_2y = \frac{M_bx^2}{2} + \frac{S_{b+1}x^3}{6} + t_{b+1}E_2I_2x + E_2I_2h_b \tag{5}$$

From Eq. (2), by integration,

$$E_2I_2\frac{dy}{dx} = M_bx + S_{b+1}\frac{x^2}{2} - P_2\frac{x^2}{2} + P_2k_2L_2x + C_3E_2I_2 \tag{6}$$

$$E_2I_2y = M_b\frac{x^2}{2} + S_{b+1}\frac{x^3}{6} - P_2\frac{x^3}{6} + P_2k_2L_2\frac{x^2}{2} + C_3E_2I_2x + C_4E_2I_2 \tag{7}$$

When $x = k_2L_2$, $E_2I_2\frac{dy}{dx}$ in Eq. (6) = corresponding value in Eq. (3). Therefore

$$E_2I_2C_3 = t_{b+1}E_2I_2 - \frac{P_2(k_2L_2)^2}{2} \tag{8}$$

When $x = L_2$, $y = h_c$. Therefore

$$C_4E_2I_2 = E_2I_2h_c - M_b\frac{L_2^2}{2} - S_{b+1}\frac{L_2^3}{6} + \frac{P_2L_2^3}{6} - \frac{P_2k_2L_2^3}{2} - t_{b+1}E_2I_2L_2 + \frac{P_2k_2^2L_2^3}{2}$$

Hence

$$E_2I_2y = \frac{M_b}{2}(x^2 - L_2^2) + \frac{S_{b+1} - P_2}{6}(x^3 - L_2^3) + \frac{P_2}{2}(k_2L_2x)(x - k_2L_2) + t_{b+1}E_2I_2(x - L_2) + E_2I_2h_c + \frac{P_2}{2}k_2L_2^3(k_2 - 1) \tag{9}$$

The value of y given by Eq. (5) equals its value as given by Eq. (9) when $x = k_2L_2$; equating these values gives

$$0 = -\frac{M_bL_2^2}{2} - \frac{S_{b+1}L_2^3}{6} - \frac{P_2L_2^3}{6}(k_2^3 - 1) - t_{b+1}E_2I_2L_2 + E_2I_2(h_c - h_b) + \frac{P_2k_2L_2^3(k_2 - 1)}{2}$$

Hence

$$t_{b+1}E_2I_2 = \frac{P_2L_2^2}{6}(1 - 3k_2 + 3k_2^2 - k_2^3) - \frac{M_bL_2}{2} - \frac{S_{b+1}}{6}L_2^2 + \frac{E_2I_2}{L_2}(h_c - h_b)$$

But

$$M_c = M_b + S_{b+1}L_2 - P_2L_2(1 - k_2) \tag{10}$$

Therefore, by substituting for S_{b+1}, its value from Eq. (10), we obtain

$$t_{b+1} = \frac{L_2}{6E_2I_2}[P_2L_2(1 - k_2)(k_2 - 2)k_2 - 2M_b - M_c] + \frac{1}{L_2}(h_c - h_b) \tag{11}$$

When $x = L_2$, $dy/dx = t_c$; therefore the value of t_c may be obtained from Eq. (6) by placing $x = L_2$ and substituting for C_3 its value as obtained from Eq. (8) after substituting for t_{b+1} its value from Eq. (11) and substituting for S_{b+1} its value from Eq. (10). The result thus obtained is given by the following equation:

$$t_c = \frac{L_2}{6E_2I_2}[M_b + 2M_c + P_2L_2k_2(1 - k_2^2)] + \frac{1}{L_2}(h_c - h_b) \tag{12}$$

By similar operations in span L_1 an equation identical with Eq. (12) would be obtained with the indices reduced to correspond to the nomenclature of span L_1. The equation for this span may therefore be written at once and will be as follows:

$$t_{b-1} = \frac{L_1}{6E_1I_1}[M_a + 2M_b + P_1L_1k_1(1 - k_1^2)] + \frac{1}{L_1}(h_b - h_a) \tag{13}$$

The two values, t_{b+1} given by Eq. (11) and t_{b-1} given by Eq. (13), are identical, since they equal the tangents to the slope of the neutral axis at two sections an infinitesimal distance apart; hence they may be placed equal to each other, thereby enabling the following equation to be written:

$$\frac{L_2}{6E_2I_2}[P_2L_2(1 - k_2)(k_2 - 2)k_2 - 2M_b - M_c] + \frac{1}{L_2}(h_c - h_b)$$
$$= \frac{L_1}{6E_1I_1}[M_a + 2M_b + P_1L_1k_1(1 - k_1^2)] + \frac{1}{L_1}(h_b - h_a)$$

Therefore

$$\frac{L_1}{E_1I_1}[M_a + 2M_b] + \frac{L_2}{E_2I_2}[2M_b + M_c] = \frac{P_1L_1^2}{E_1I_1}(k_1^3 - k_1)$$
$$+ \frac{P_2L_2^2}{E_2I_2}(3k_2^2 - k_3^3 - 2k_2) + 6\left[\frac{(h_c - h_b)}{L_2} + \frac{(h_a - h_b)}{L_1}\right] \tag{14}$$

If the axis, from which h_a, h_b, and h_c are measured, is taken as the unstressed axis of the beam, h_a, h_b, and h_c may be replaced by δ_a, δ_b, and δ_c, respectively. Then Eq. (14) becomes

$$\frac{L_1}{E_1I_1}[M_a + 2M_b] + \frac{L_2}{E_2I_2}[2M_b + M_c] = \frac{P_1L_1^2}{E_1I_1}(k_1^3 - k_1) + \frac{P_2L_2^2}{E_2I_2}(3k_2^2 - k_2^3 - 2k_2) + 6\left[\frac{(\delta_a - \delta_b)}{L_1} + \frac{(\delta_c - \delta_b)}{L_2}\right] \qquad (15)$$

Equation (15) is the three-moment equation. It is applicable to structures in which the product of the moment of inertia and modulus of elasticity is constant within each span. The sign conventions in this equation should be noted carefully. Bending moments are positive when causing compression in the upper fibres of the beam. All values should be taken in the same units. Downward loads are positive. The support deflections δ are positive when upward and negative when downward.

To obtain the corresponding equation for uniform loads, substitute the following values, and integrate between proper limits:

$$P_1 = w_1dx_1, \qquad P_2 = w_2dx_2, \qquad k_1L_1 = x_1, \qquad \text{and} \qquad k_2L_2 = x_2$$

In these two equations w_1 and w_2 are the loads per foot on the two spans.

For the case where the load extends over the entire length of one or more spans and is uniform over each span to which it is applied this gives:

$$\frac{L_1}{E_1I_1}[M_a + 2M_b] + \frac{L_2}{E_2I_2}[2M_b + M_c] = \frac{L_1^2}{E_1I_1}\int_0^{L_1} w_1dx_1 \left[\left(\frac{x_1}{L_1}\right)^3 - \frac{x_1}{L_1}\right] + \frac{L_2^2}{E_2I_2}\int^{L_2} w_2dx_2\left[3\left(\frac{x_2}{L_2}\right)^2 - \left(\frac{x_2}{L_2}\right)^3 - 2\left(\frac{x_2}{L_2}\right)\right] + 6\left[\frac{(\delta_a - \delta_b)}{L_1} + \frac{(\delta_c - \delta_b)}{L_2}\right] = -\frac{w_1L_1^3}{4E_1I_1} - \frac{w_2L_2^3}{4E_2I_2} + 6\left[\frac{(\delta_a - \delta_b)}{L_1} + \frac{(\delta_c - \delta_b)}{L_2}\right] \qquad (16)$$

If the unstressed axis is either straight or slightly curved before the loads are applied, and if the supports fit the curve of this unstressed girder and are unyielding, the application of either Eq. (15) or Eq. (16) will give the correct results by placing

$\delta_a - \delta_b = 0$ and $\delta_c - \delta_b = 0$. If a point of support settles under load, or if the unstressed beam does not fit the supports, then δ_a, δ_b, and δ_c should be replaced by their appropriate values.

Beams with fixed ends, *i.e.*, ends so constructed that the change of slope under load is zero, may be computed by assuming that a span of zero length may be added at each of the fixed ends and then applying the three-moment equation to the hypothetical case.

4. Applications of the Three-moment Equation.—For purpose of illustration the following examples of the application of the three-moment equation are given:

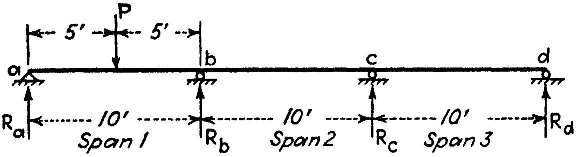


FIG. 3.—Constant EI. $E = 30 \times 10^6$ lb. per square inch; $I = 300$ in.^{4}.

Problem 1: Compute by the three-moment equation the girder reactions for the load shown in Fig. 3 if the unloaded girder fits the supports, and the latter are unyielding.

Solution: Apply Eq. (15) to spans 1 and 2, noting that for this case values of L and M may be taken in foot-pounds, since the values of EI cancel.

$$\delta_a = \delta_b = \delta_c = 0, \qquad M_a = 0, \qquad P_1 = P, \qquad P_2 = 0, \qquad k_1 = 0.5$$

$$\frac{10}{EI}[0 + 2M_b] + \frac{10}{EI}[M_c + 2M_b] = \frac{P(10)^2}{EI}[(0.5)^3 - (0.5)]$$

$$4M_b + M_c = -3.75P \tag{a}$$

Apply Eq. (15) to spans 2 and 3:

$$M_d = 0, \qquad P_1 = 0, \qquad P_2 = 0$$

$$\frac{10}{EI}[M_b + 2M_c] + \frac{10}{EI}[0 + 2M_c] = 0$$

$$M_b + 4M_c = 0 \tag{b}$$

Solving Eqs. (*a*) and (*b*) simultaneously,

$$M_c = +\frac{3.75P}{15} \text{ ft.-lb.} \qquad M_b = -P \text{ ft.-lb.}$$

Using these values and applying the equations of statics give the vertical reactions as follows:

$$R_a = \frac{16}{40}P \uparrow \qquad R_b = \frac{29}{40}P \uparrow \qquad R_c = \frac{6}{40}P \downarrow \qquad R_d = \frac{P}{40} \uparrow$$

Problem 2: Compute by the three-moment equation the girder reactions when the load shown in Fig. 3 equals 25,000 lb. if the unloaded girder fits the supports, but support b yields 0.1 in. under load.

Solution: Apply Eq. (15) to spans 1 and 2.

$$\delta_a = 0,\ \delta_c = 0,\ \delta_b = -0.1 \text{ in.},\ M_a = 0, P_1 = 25{,}000 \text{ lb.}, P_2 = 0,\ k_1 = 0.5$$

$$\frac{10}{EI}[0 + 2M_b] + \frac{10}{EI}[M_c + 2M_b] = \frac{(25{,}000)(10)^2}{EI}[(0.5)^3 - (0.5)] + 6\left[\left(0 - \frac{-0.1}{12}\right)\left(\frac{1}{10}\right) + \left(0 - \frac{-0.1}{12}\right)\left(\frac{1}{10}\right)\right]$$

$$4M_b + M_c = -93{,}750 + \frac{6EI(0.2)}{(10)^3(12)} = -93{,}750 + \frac{(6)(30 \times 10^6)(12^2)(300)(0.2)}{(10)^3\quad(12)\quad(12)^4}$$

from which

$$4M_b + M_c = -31{,}250 \qquad (a)$$

Apply Eq. (15) to spans 2 and 3.

$$\delta_b = -0.1 \text{ in.}, \qquad \delta_c = \delta_d = 0, \qquad M_d = 0, \qquad P_1 = 0, \qquad P_2 = 0.$$

$$\frac{10}{EI}[M_b + 2M_c] + \frac{10}{EI}[0 + 2M_c] = 6\left[\left(\frac{-0.1}{12} - 0\right)\frac{1}{10} + \left(\frac{0 - 0}{10}\right)\right]$$

$$M_b + 4M_c = -\frac{6EI(0.1)}{(10)^2(12)} = \frac{-(6)(30 \times 10^6)(12)^2(300)(0.1)}{(10)^2\quad(12)\quad(12)^4}$$

from which

$$M_b + 4M_c = -31{,}250 \qquad (b)$$

Solving Eqs. (a) and (b) simultaneously,

$$M_b = -6250 \text{ ft.-lb.} \qquad M_c = -6250 \text{ ft.-lb.}$$

Using these values, the vertical reactions are

R_a = 11,875 lb. ↑ R_b = 13,125 lb. ↑ R_c = 625 lb. ↑ R_d = 625 lb. ↓

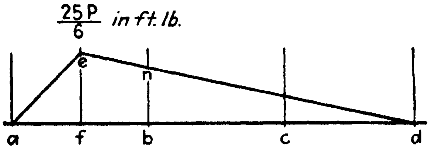
It should be noticed that extreme care must be used in keeping the units consistent and substituting the correct signs of the Δ terms.

5. Analytical Solution of Continuous Girder by the Moment-area Method.—The moment-area method of deflections may be readily adapted to the solution of continuous girders having constant moments of inertia and moduli of elasticity by determining the deflections in terms of the applied loads and the reactions and solving as in the previous example. This method is illustrated by the following problems.

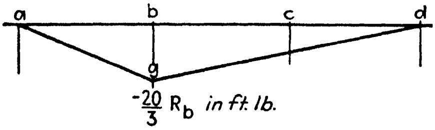
Problem 1: Determine the reactions for the continuous beam used in Prob. 1 of Art. 4.

Solution: Construct moment curves for the beam assuming it to be supported at ends only and subjected to three forces applied between ends, *viz.*, P, R_b, and R_c.

The moment curves for P and R_b are shown in Fig. 4. Since the structure is symmetrical, the moment curve for R_c is identical with that for R_b, turned end for end.



Moment Curve for Applied Load P



Moment Curve for Reaction R_b

FIG. 4.

Since the vertical deflection at any point on the axis of an end-supported beam equals the moment at the section containing the point in question due to hypothetical forces equal to the moment areas divided by EI,[1] we can now write expressions for the deflections at B and C due to the various forces in terms of the moment areas.

$$\text{Moment area } aef = \left(\frac{25}{6}P\right)\left(\frac{5}{2}\right) = \frac{125}{12}P \text{ lb. ft.}^2$$

$$\text{Moment area } efd = \left(\frac{25}{6}P\right)\left(\frac{25}{2}\right) = \frac{625}{12}P \text{ lb. ft.}^2$$

$$\text{Moment area } bnd = \frac{4}{5}\left(\frac{25}{6}P\right)(10) = \frac{100}{3}P \text{ lb. ft.}^2$$

$$\text{Moment area } abg = \left(\frac{20}{3}R_b\right)\left(\frac{10}{2}\right) = \frac{100}{3}R_b \text{ lb. ft.}^2$$

$$\text{Moment area } bgd = \left(\frac{20}{3}R_b\right)\left(\frac{20}{2}\right) = \frac{200}{3}R_b \text{ lb. ft.}^2$$

Reactions due to moment areas:

$$\text{Due to } P \begin{cases} R_a = \dfrac{125}{12}P\left(\dfrac{25+\frac{5}{3}}{30}\right) + \dfrac{625}{12}P \times \dfrac{50}{3 \times 30} = \dfrac{4125P}{108} \text{ lb. ft.}^2 \\ R_d = \left[\dfrac{125+625}{12} - \dfrac{4125}{108}\right]P = \dfrac{2625P}{108} \text{ lb. ft.}^2 \end{cases}$$

---

[1] See Theory of Structures, 3d ed., Art. 185, Spofford.

Due to R_b

$$R_a = -\left[\frac{100}{3}R_b\left(\frac{20+\frac{10}{3}}{30}\right) + \frac{200}{3}R_b \times \frac{20 \times 2}{3 \times 30}\right] = -\frac{500}{9}R_b \text{ lb. ft.}$$ [3]

$$R_d = -\left[\left(\frac{100}{3} + \frac{200}{3}\right)R_b - \frac{500}{9}R_b\right] = -\frac{400}{9}R_b \text{ lb. ft.}$$ [3]

Due to R_c

$$R_a = -\frac{400}{9}R_c \text{ lb. ft.}$$ [3]

$$R_d = -\frac{500}{9}R_c \text{ lb. ft.}$$ [3]

To obtain deflections write expressions for moments at b and c giving following results, the moment values being in units of lb. ft.[3]

$$EI\delta_b = \frac{2625P}{108} \times 20 - \frac{100}{3}P \times \frac{20}{3} - \frac{500}{9}R_b \times 10 + \frac{100}{3}R_b \times \frac{10}{3} - \frac{400}{9}R_c \times 10 + \frac{200}{3}R_c\left(\frac{1}{4}\right)\left(\frac{10}{3}\right)$$

$$= \frac{28{,}500P}{108} - \frac{4000}{9}R_b - \frac{3500}{9}R_c$$

To get value of δ_b in feet, E and I should be in foot units. Therefore multiply E in pounds per square inch by $(12)^2$, and divide I in in.^{4} by $(12)^4$.

$$EI\delta_c = \frac{2625P}{108} \times 10 - \frac{100}{3} \times \frac{1}{4} \times \frac{10}{3}P - \frac{4000}{9}R_c - \frac{3500}{9}R_b$$

$$= \frac{23{,}250P}{108} - \frac{4000}{9}R_c - \frac{3500}{9}R_b$$

If the supports do not settle, $\delta_b = 0$ and $\delta_c = 0$; hence the foregoing equations may be written as follows:

$$28.5P - 48R_b - 42R_c = 0$$
$$23.25P - 48R_c - 42R_b = 0$$

These may be written as follows:

$$19P - 32R_b - 28R_c = 0$$
$$31P - 56R_b - 64R_c = 0$$

When the last two equations are solved simultaneously, the following results which agree with those previously found by the three-moment equation are obtained:

$$R_b = \frac{29}{40}P \uparrow \qquad R_c = \frac{6}{40}P \downarrow$$

The remaining reactions may now be solved by statics.

Problem 2: Solve Prob. 2 of Art. 4 by the moment-area method.

Solution: In this problem $\delta_b = -0.1$ in., $\delta_c = 0$. So the equations for $EI\delta_b$ and $EI\delta_c$ in the previous problem become

$$EI\delta_b = \frac{(28{,}500)(25{,}000)}{108} - \frac{4000}{9}R_b - \frac{3500}{9}R_c = + \left(\frac{0.1}{12}\right)(30 \times 10^6)(12)^2\left(\frac{300}{(12)^4}\right)$$

$$EI\delta_c = \frac{(23{,}250)(25{,}000)}{108} - \frac{3500R_b}{9} - \frac{4000}{9}R_c = 0$$

These equations become

$$4R_b + 3.5R_c = +54{,}687.5$$
$$3.5R_b + 4R_c = +48{,}437.5$$

which, when solved simultaneously, give the following results which also agree with the values found by the three-moment equation:

$$R_b = 13{,}125 \text{ lb.} \uparrow \qquad R_c = 625 \text{ lb.} \uparrow$$

The other reactions may be found by statics.

In solving problems where the deflection is a factor it is necessary to use consistent units. In the foregoing examples the moment areas were taken in foot-pound units; hence it was necessary to reduce the deflections to foot units, E to pounds per square foot and I to foot units. The reader is advised, however, in such problems to use inch units throughout to avoid confusion.

In addition to exercising care in keeping the units consistent, the signs of the various quantities must be properly interpreted. In this problem the sign of δ_b is determined by the convention adopted in writing the equation for $EI\delta_b$ in Prob. 1 of this article; *i.e.*, δ_b is positive when downward.

If, instead of having a constant value of EI throughout its length, the girder had either a varying moment of inertia in any span or a different moment of inertia in the separate spans, or varying values of E, it would have been necessary to work with the M/EI curve instead of the M curve in applying the moment-area theorems. For cases of this nature the graphical method given in the following article may be used advantageously.

6. Graphical Solution of Continuous Girder by the Moment-area Method.—The moment-area method of deflections may be applied to a beam of varying cross section by dividing the moment area into sections each having a length such that the portion of the beam subtended by the section has a constant or approximately constant value of EI. If the value of EI is changing rapidly, as in the haunches of a reinforced-concrete beam, the sections for this portion of the beam should be so short that no material error would result by using the average value of EI for each section.

Deflections for an end-supported beam by the moment-area method may be obtained graphically by the following method:

1. Plot a funicular polygon for the beam due to the given loads. It is a well-known principle of graphic statics that the ordinate at any point on the axis of the beam between the polygon and the closing line equals the moment at the section containing the point to some scale.

2. Divide the area between the closing line corresponding to the preceding funicular polygon and the polygon into areas of convenient size and shape such that their combined areas will agree closely with the total area under consideration. These areas should also be of such shapes that their centroids can be readily located and of such length that EI can be considered as constant over the section length.

3. Draw another funicular polygon using as forces the actual areas in square inches determined under (2) reduced to some convenient common value of EI and assuming these forces to act vertically through their respective centroids.

The ordinates intercepted between the closing line and the funicular polygon constructed under (3) represent the vertical deflections at all points along the beam. The funicular polygon will be the actual curve of deflections to some scale if constructed so that the closing line coincides with the beam axis.[1] If the closing line does not coincide with the beam axis, the curve of deflections may be constructed by plotting the various ordinates intercepted between the closing line and the funicular polygon upon the beam axis.

To determine the actual deflection, the scales to be used must be given careful consideration. Under (1), the bending moment at any section equals the ordinate to scale of distance multiplied by H to scale of force. If scale of distance equals d in. per inch, and scale of force equals f lb. per inch, and if ordinates are represented in inches by y, the moment at any point in inch-pounds equals $ydfH$, and the moment area per inch of length of diagram equals yd^2fH.

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[1] See Theory of Structures, 3d ed., Chap. XIII, Spofford.

If under (3) the force diagram corresponding to moment areas is plotted to a scale of m sq. in. per inch, and the pole distance in inches is H_1, the pole distance to the scale used will equal md^2fHH_1. If the beam be plotted again to a scale of d in. per inch, and z equals ordinate in inches at any point between the funicular polygon constructed for forces corresponding to the moment area and its closing line, the actual deflection of δ in inches at that point will be given by the expression

$$\delta = \frac{(zd)(md^2fHH_1)}{EI} = \frac{mzd^3fHH_1}{EI} \tag{17}$$

The foregoing method is based upon the fact that the moment is the second integral of the load, and the deflection is the second integral of the moment divided by EI; hence, the relation between moment and load is similar to that between deflection and moment. It should be noted that both moment and deflection at ends of beam equal zero; hence, constants of integration disappear.

The following problem shows the application of this method to a uniformly loaded beam of constant cross section. For a beam of variable section the value of each moment area should be corrected by adopting the moment area for any convenient section of the beam as a standard and multiplying it by the ratio between EI of the standard section and EI of the section under consideration. The deflection will then be given by Eq. (17) in which E and I are the values of the standard section.

Problem Illustrating Graphical Method: Construct curve of deflections for an end-supported horizontal beam of constant EI loaded over entire length with a vertical load of 1000 lb. per foot.

NOTE: This loading chosen for clearness of presentation, but method equally applicable to concentrated loads or to uniform load over part of span only or to any other form of loading.

Solution (see Fig. 5): 1. Lay off half of beam to scale d in. = 1 in., in this case, 96 in. = 1 in., and subdivide the uniform load into forces acting sufficiently near together to give results with necessary degree of precision. In this particular case the loading diagram is subdivided into 3-ft. lengths.

2. Plot a force polygon for preceding loads to a convenient scale. In this case, 12,000 lb. = 1 in.

3. Construct the funicular polygon shown in the upper portion of Fig. 5.

4. Divide the area between the funicular polygon and the beam axis into the triangles and rectangles shown by capital letters, and determine their areas in square inches.

5. Draw vertical lines through centroids of these areas.

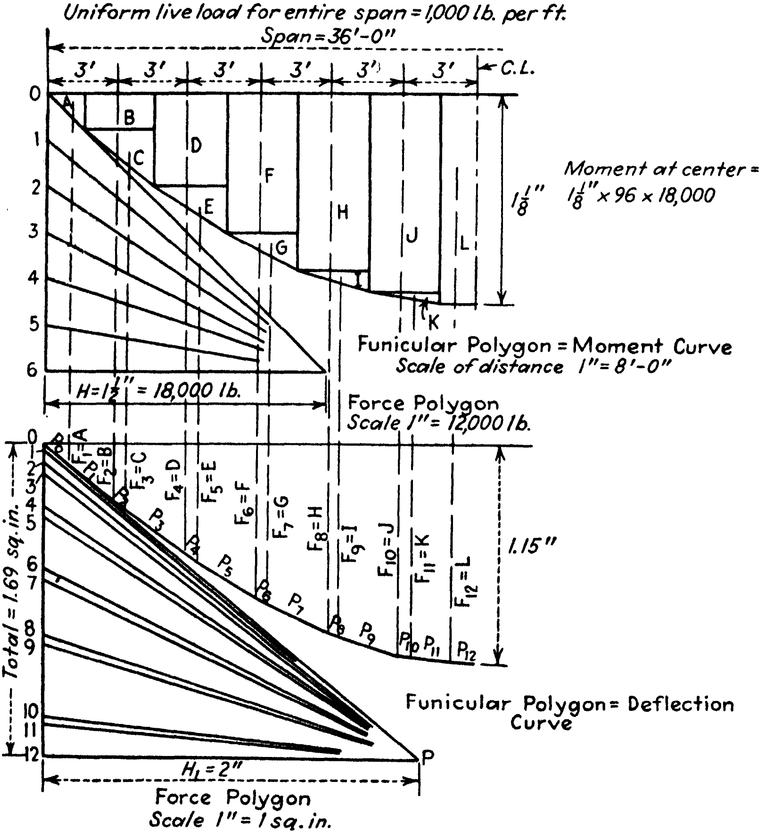

FIG. 5.

6. Construct a new force polygon, using these areas for forces, to a scale of m sq. in. = 1 in.; in this case, 1 sq. in. = 1 in., and draw a new funicular polygon using the same scale of distance as before, *viz.*, 96 in. = 1 in.

NOTE: By taking advantage of symmetry in this problem, it is easy to select the position of the poles of the force polygons so that the closing lines of the funicular polygons are horizontal.

This new curve represents the curve of deflections, and the actual value of the deflection at any point may be obtained by applying Eq. (17).

For the case shown, $z = 1.15$ in., $m = 1$, $d = 96$, $f = 12{,}000$, $H = 1\frac{1}{2}$ in., and $H_1 = 2$ in. Hence, the centre deflection is given by the following expression:

$$\delta_{cg} = \frac{1 \times 1.15 \times 96 \times 96 \times 96 \times 12{,}000 \times 1.5 \times 2}{EI}$$

By analytical methods, the deflection at centre for the previous problem is given by the following expression:

$$\delta_{ca} = \frac{5}{384}\,\frac{1000}{12} \times \frac{36 \times 36 \times 36 \times 36 \times 12 \times 12 \times 12 \times 12}{EI}$$

It follows that

$$\frac{\delta_{cg}}{\delta_{ca}} = \frac{1 \times 1.15 \times 96 \times 96 \times 96 \times 12{,}000 \times 1.5 \times 2}{\frac{5}{384} \times 1000 \times 36 \times 36 \times 36 \times 36 \times 12 \times 12 \times 12}$$

$$= \frac{97}{100}$$

Evidently the graphical method gives a result for the deflection as precise as could be expected from the precision possible with graphical methods. The same process may be used for determining the deflections due to a concentrated load.

In applying this method to continuous girders, the girder is considered as being end supported and subjected to the applied loads and the unknown interior reactions. By this method, the deflection at each of the interior reaction points may be expressed in terms of the applied loads and the unknown interior reactions. Once the deflections are known, the reactions of the continuous girder may be solved as in the analytical method of moment areas.

7. Continuous Girders. Reactions by Analytical Method of Deflections.—Reactions of continuous girders either fixed or supported at the ends may also be computed by assuming the girder to be supported at the ends only or at any other two points of support and equating the deflections at points of application of the other reactions due to the applied loads, with the deflections due to the unknown reactions, the expressions for the deflections in all cases being expressed by analytical methods. The expressions for rotations at fixed ends must also be placed equal to zero.

Consider a continuous beam such as that shown in Fig. 6 with $n + 1$ supports, fixed against rotation at the end supports, and with vertical reactions numbered as shown.

This structure is statically indeterminate under vertical loads to the $[(n - 1) + 2]$ degree; *i.e.*, there are $[(n - 1) + 2]$ reactions and moments which cannot be determined by statics. The necessary equations for their solution may be obtained by placing the expression for the deflection at each intermediate support equal to its deflection from the unloaded position of the girder, which if the supports are unyielding is equal to zero, and the expressions for the rotation at each end equal to the rotation of that end which will be zero if the support is unyielding.

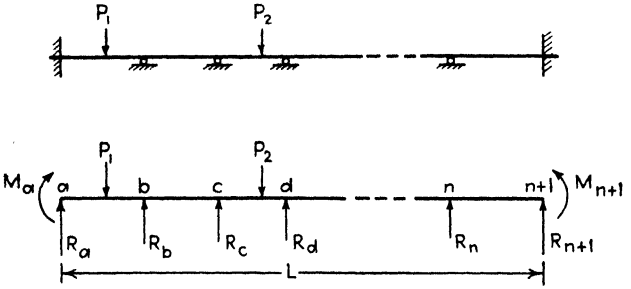

FIG. 6.

Let M = moment at any section of beam.

M_0 = moment at any section due to applied loads, assuming beam supported only at the ends, and positive when causing compression in upper fibres.

m_b, m_c, m_d, etc. = moment at any section due to upward forces of unity at b, c, d, etc., assuming beam supported at ends only; positive when causing compression in upper fibres.

m_a, and m_{n+1} = moment at any section due to a unit moment acting in direction of M_a and M_{n+1}, respectively; positive when causing compression in upper fibres.

It follows that

$$M = M_0 + m_aM_a + m_{n+1}M_{n+1} + m_bR_b + m_cR_c + \cdots + m_nR_n$$

Let δ_b, δ_c, etc. = upward deflection of points b, c, etc.

α_a = rotation of point a in direction of M_a.

α_{n+1} = rotation of point $n + 1$ in direction of M_{n+1}.

Now, by applying the equations δ or $\alpha = \int \frac{Mmdx}{EI}$[*], we may write $[(n - 1) + 2]$ independent equations, of which the following are typical, and determine the unknown reactions by their simultaneous solution. Evidently if all the supports are unyielding, δ_a, δ_b, etc., should be placed equal to zero; otherwise the amount of settlement must be used in the solutions.

Therefore,

$$\left.\begin{aligned}
\alpha_a &= \int (M_0 + m_a M_a + m_{n+1} M_{n+1} + m_b R_b + m_c R_c + \cdots \\
&\qquad + m_n R_n)(m_a)\frac{dx}{EI}\dagger \\
\delta_b &= \int (M_0 + m_a M_a + m_{n+1} M_{n+1} + m_b R_b + m_c R_c + \cdots \\
&\qquad + m_n R_n)(m_b)\frac{dx}{EI}\dagger \\
\delta_n &= \int (M_0 + m_a M_a + m_{n+1} M_{n+1} + m_b R_b + m_c R_c + \cdots \\
&\qquad + m_n R_n)(m_n)\frac{dx}{EI}\dagger
\end{aligned}\right\} \tag{18}$$

It is also possible to apply this method to a girder of varying cross section, in which case the integration of any portion of the girder in which the moment of inertia is not constant must be divided into successive steps, each covering a portion of the girder.

As given here, no account has been taken of shear distortion, which in the ordinary case is so small as to be negligible, although, if it were deemed advisable, it could easily be included by use of the equation

$$\delta = \int \frac{Mmdx}{EI} + \int \frac{Vvdx}{A'G}$$

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[*] See Theory of Structures, 3d ed., Arts. 177 and 178, Spofford.

[†] Since the internal work in a girder equals $W = \int \frac{M^2dx}{2EI}$, the foregoing expressions may be obtained by differentiating the expression for work with respect to the different variables as follows:

$$\frac{\partial W}{\partial M_a} = \alpha_a = \int Mm_a \frac{dx}{EI}$$

$$\frac{\partial W}{\partial R_b} = \delta_b = \int Mm_b \frac{dx}{EI}, \text{ etc.}$$

These expressions are identical with those obtained by application of Castigliano's law.

In the second term of the preceding equation,

V = external shear at any section due to the applied loads and reactions.

v = external shear at any section due to unit load, applied at point of and in direction of desired deflection.

A' = cross-sectional area over which shear may be assumed uniformly distributed.

G = modulus of material in shear.

8. Application of the Analytical Method of Deflections to Determination of Girder Reactions.—The application of this method is illustrated by the following example:

Problem 1: Determine by the analytical method of deflections the reactions on the beam shown in Fig. 3 due to applied load P.

Solution: Application of Eq. (18) gives

$$\begin{aligned} EI\delta_b = {} & \int_0^5 \left(\frac{5Px}{6}\right)\left(\frac{-2x}{3}\right)dx + \int_5^{10}\left[\frac{5Px}{6} - P(x-5)\right]\left(\frac{-2x}{3}\right)dx \\ & + \int_0^{20}\left(\frac{Px}{6}\right)\left(\frac{-x}{3}\right)dx + \int_0^{10} R_b\left(\frac{-2x}{3}\right)^2 dx + \int_0^{20} R_b\left(\frac{-x}{3}\right)^2 dx \\ & + \int_0^{10} R_c\left(\frac{-2x}{3}\right)\left(\frac{-x}{3}\right)dx + \int_{10}^{20} R_c\left(\frac{-x}{3}\right)\left[\frac{-2x}{3} + (x-10)\right]dx \\ & + \int_0^{10} R_c\left(\frac{-x}{3}\right)\left(\frac{-2x}{3}\right)dx \end{aligned}$$

$$\begin{aligned} EI\delta_c = {} & \int_0^5 \left(\frac{5Px}{6}\right)\left(\frac{-x}{3}\right)dx + \int_5^{20}\left[\frac{5Px}{6} - P(x-5)\right]\left(\frac{-x}{3}\right)dx \\ & + \int_0^{10}\left(\frac{Px}{6}\right)\left(\frac{-2x}{3}\right)dx + \int_0^{10} R_b\left(\frac{-2x}{3}\right)\left(\frac{-x}{3}\right)dx + \int_{10}^{20} R_b\left[\frac{-2x}{3}\right. \\ & \left. + (x-10)\right]\left(\frac{-x}{3}\right)dx + \int_0^{10} R_b\left(\frac{-x}{3}\right)\left(\frac{-2x}{3}\right)dx + \int_0^{10} R_c\left(\frac{-2x}{3}\right)^2 dx \\ & + \int_0^{20} R_c\left(\frac{-x}{3}\right)^2 dx \end{aligned}$$

Performing the indicated integrations and inserting the various limits gives the following expressions

$$\frac{108EI\delta_b}{1000} = -28.5P + 48R_b + 42R_c = 0$$

$$\frac{108EI\delta_c}{1000} = -23.25P + 42R_b + 48R_c = 0$$

These may be written for the cases where δ_b and δ_c each equal zero:

$$-19P + 32R_b + 28R_c = 0$$

$$-31P + 56R_b + 64R_c = 0$$

Solving these simultaneously gives the following values:

$$R_b = \frac{29}{40}P \uparrow \qquad R_c = \frac{6}{40}P \downarrow$$

Problem 2: Solve Prob. 2 of Art. 4 by the method of deflections.

Solution: In this problem $\delta_b = -0.1$ in., $\delta_c = 0$, $P_1 = 25{,}000$ lb. From the equations in the foregoing problem,

$$\frac{108EI}{1000}\left(\frac{-0.1}{12}\right) = -28.5(25{,}000) + 48R_b + 42R_c$$
$$\frac{108EI}{1000}(0) = -23.25(25{,}000) + 42R_b + 48R_c$$

or

$$\frac{108(30 \times 10^6)(12)^2(300)}{(1000)(12)^4\quad(12)}\left(\frac{-0.1}{12}\right) = -2.375(25{,}000) + 4R_b + 3.5R_c$$
$$0 = -1.9375(25{,}000) + 3.5R_b + 4R_c$$

From which

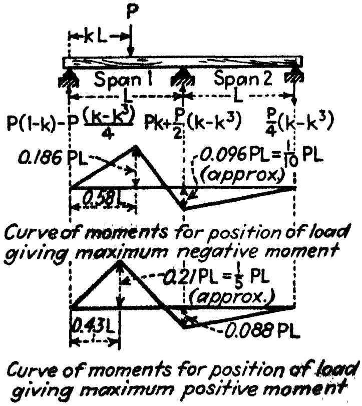
$$4R_b + 3.5R_c = +54{,}687.5$$
$$3.5R_b + 4R_c = +48{,}437.5$$

which, when solved simultaneously, give the following results which are identical with the values previously determined:

$$R_b = 13{,}125 \text{ lb.} \uparrow \qquad R_c = 625 \text{ lb.} \uparrow$$

Again, it should be noticed that the signs of the δ terms and the units used n the equations are very important.

9. Reactions, Shears, and Moments for Common Cases of Continuous Girders.—In order to simplify the determination of reactions, shears, and moments for certain common cases of continuous girders, the diagrams of Figs. 7 to 15 inclusive have been prepared. Inspection of these diagrams shows that for a continuous girder of either two or three equal spans loaded with a uniform live load w per foot the maximum live moment occurs at a support and is negative, its value equaling, for the two-span girder, that of the positive live moment on an end-supported span and being slightly less than this for the three-span girder. The maximum positive moment equals $\frac{1}{10}wL^2$ for both cases, or about three-quarters of the value that it



Curve of moments for position of load giving maximum negative moment

Curve of moments for position of load giving maximum positive moment

FIG. 7.—Continuous girder. Reactions and moments for a single concentrated load.

would have for an end-supported span. The maximum shear equals $^{37}\!/_{60}wL$ which is slightly greater than for an end-supported span. In this article the term EI is constant.

For cases where the spans are greater in number than those shown or are of varying lengths, the various functions may be obtained by the student by one of the methods explained in this chapter. Reference may also be made to tables of continuous girders such as "Kontinuierliche Träger Tabellen" by Griot (printed by Aschmann and Scheller, Zurich, Switzerland). In these tables, influence data are given for moments at sections located at frequent intervals throughout the length of continuous beams having two spans of varying ratios of length and of symmetrical continuous beams having three and four spans, the length of the centre span or spans varying from one to two times that of the end span. A series of influence lines for reactions, shears, and moments at frequent intervals throughout the length

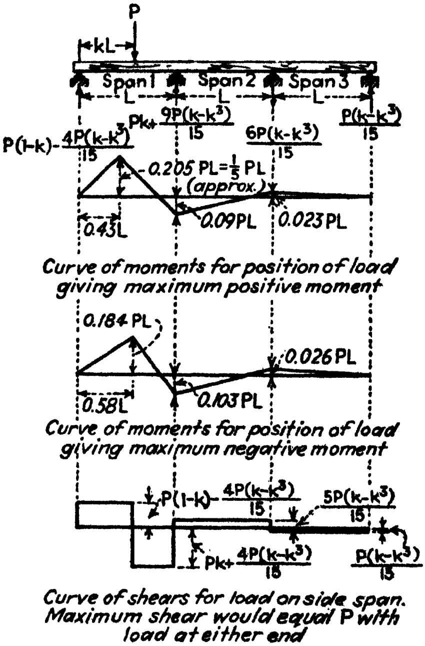

P
kL
Span 1
Span 2
Span 3
L
L
L
P(1-k)-4P(k-k³)/15
Pk+9P(k-k³)/15
6P(k-k³)/15
P(k-k³)/15
-0.205 PL=1/5 PL (approx.)
0.43L
0.09PL
0.023PL
Curve of moments for position of load giving maximum positive moment
0.184 PL
0.026PL
0.58L
0.103PL
Curve of moments for position of load giving maximum negative moment
P(1-k)-4P(k-k³)/15
5P(k-k³)/15
Pk+4P(k-k³)/15
P(k-k³)/15
Curve of shears for load on side span. Maximum shear would equal P with load at either end

FIG. 8.—Continuous girder. Curve of shears and moments. Single concentrated load.

of continuous beams of two and three equal spans is also given in "Concrete Engineer's Handbook," by Hool, published by the McGraw-Hill Book Company, Inc., New York, 1927–1928. For other data upon continuous beams see Carnegie Pocket Companion, 24th ed., pp. 156 *et seq.*

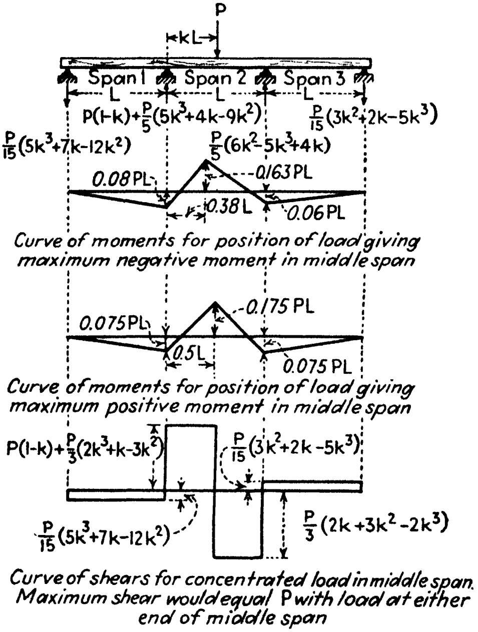

P
kL
Span 1
Span 2
Span 3
L
L
L
$P(1-k)+\frac{P}{5}(5k^3+4k-9k^2)$
$\frac{P}{15}(3k^2+2k-5k^3)$
$\frac{P}{15}(5k^3+7k-12k^2)$
$\frac{P}{5}(6k^2-5k^3+4k)$
0.08PL
-0.163PL
0.38L
0.06PL
Curve of moments for position of load giving maximum negative moment in middle span
-0.175PL
0.075PL
0.5L
0.075PL
Curve of moments for position of load giving maximum positive moment in middle span
$P(1-k)+\frac{P}{5}(2k^3+k-3k^2)$
$\frac{P}{15}(3k^2+2k-5k^3)$
$\frac{P}{15}(5k^3+7k-12k^2)$
$\frac{P}{3}(2k+3k^2-2k^3)$
Curve of shears for concentrated load in middle span. Maximum shear would equal P with load at either end of middle span

FIG. 9.—Continuous girder. Curves of moments and shears. Single concentrated load.

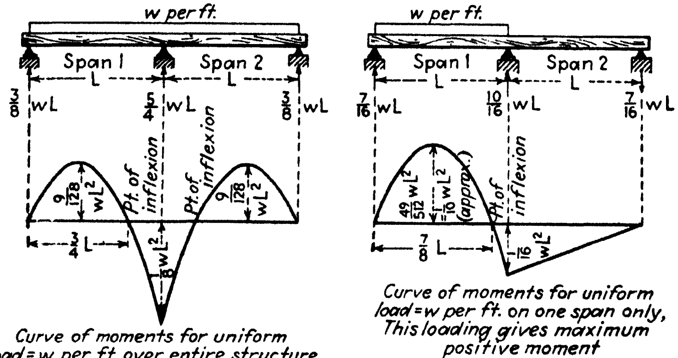
10. Continuous Trusses. Method of Design.—The design of statically indeterminate continuous trusses involves:

1. The application of approximate methods such as the three-moment equation to obtain a preliminary design.

2. The determination of the stresses in the preliminary design by the application of the method of least work or the method of deflections or such other methods as may be applicable to conditions arising in practice.

3. The revision of the design if found expedient by the results of the computation under 2.

For a truss having a constant or practically constant moment of inertia, the three-moment equation is often sufficiently precise for the final design; *e.g.*, a parallel-chord short-span truss may be computed with sufficient accuracy by this method. It should be noted that the web stresses in the centre span of a three-span

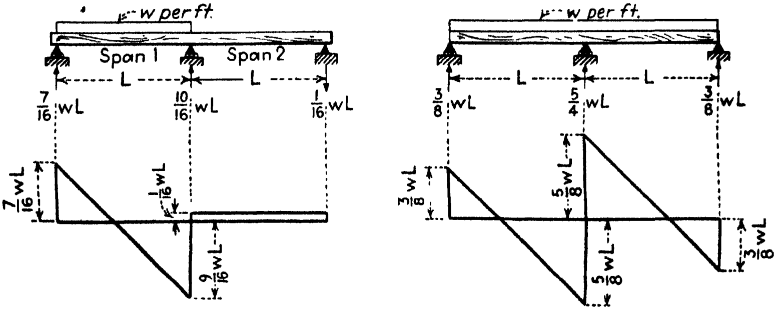


Curve of moments for uniform load = w per ft. over entire structure, This loading gives maximum negative moment

Curve of moments for uniform load = w per ft. on one span only, This loading gives maximum positive moment

FIG. 10.—Continuous girder. Curves of moment for uniform load.

symmetrical continuous truss with horizontal chords and symmetrically loaded, a loading condition which is likely to exist under dead loads, can be computed directly by statics by working from the center of the span. The dead reactions may be established by jacking the ends of the structure until the dead reactions equal those determined analytically.

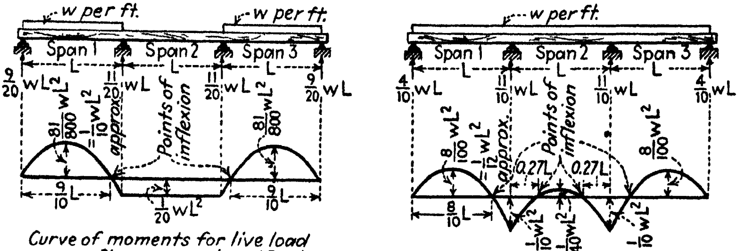


Curve of shears for uniform load = w per ft. on one span only

Curve of shears for uniform load = w per ft. over both spans This loading gives maximum shear

FIG. 11.—Continuous girder. Curves of shears for uniform load.

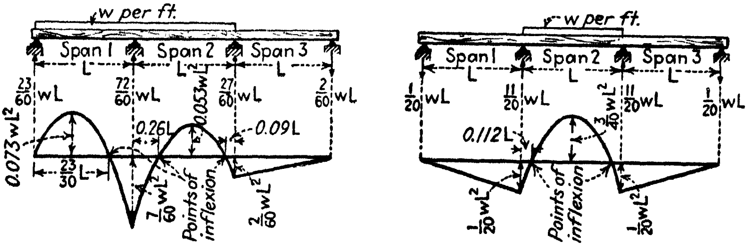
11. Continuous Trusses. Theory.—Expressions for stresses in the redundant bars of continuous trusses may be developed by either the method of least work or the method of deflections, as explained in "Theory of Structures."[1] Their development and application to a specific case—the Lake Champlain three-span continuous truss shown in the Frontispiece and in Fig. 16—follow. The analysis of two-span continuous trusses is outlined at the end of this article.



Curve of moments for live load = w per ft. on spans 1 and 3 only. This loading gives maximum positive moment on spans 1 and 3

Curve of moments for live load = w per ft. over entire girder

Fig. 12.—Continuous girder. Curves of moments for uniform load.



Curve of moments for live load = w per ft. on spans 1 and 2. This loading gives maximum negative moment on structure

Curve of moments for live load = w per ft. on span 2 only This loading gives maximum positive moment on span 2

Fig. 13.—Continuous Girder. Curves of moments for uniform loads.

The Lake Champlain Bridge is indeterminate to the second degree and it is convenient to select U_9U_{10} and $U_{24}U_{25}$ as the two redundant bars.

Let A = area of any bar in square inches.

X = stress in bar U_9U_{10} (Fig. 16), assumed tension.

Y = stress in bar $U_{24}U_{25}$ (Fig. 16), assumed tension.

P = stress in any bar due to load of unity applied at any panel point, assuming structure to act as three simple spans, *i.e.*, with bars U_9U_{10} and $U_{24}U_{25}$ cut.

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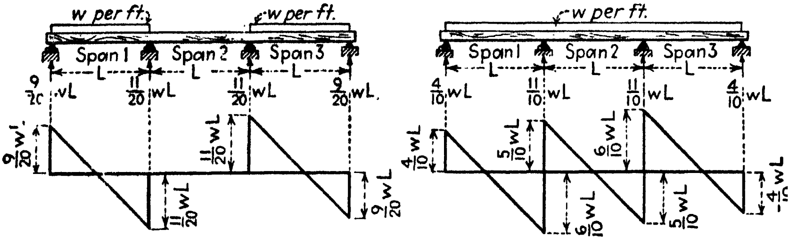
[1] 3d ed., 3d and later impressions, Spofford. McGraw-Hill Book Company, Inc.

S_1 = stress in any bar with bars U_9U_{10} and $U_{24}U_{25}$ cut, due to tension of unity in place of stress in bar U_9U_{10}.

S_2 = stress in any bar due to tension of unity in place of stress in bar $U_{24}U_{25}$.

S = actual stress in any bar, $= P + S_1X + S_2Y$.

E = modulus of elasticity in the various truss members assumed as constant throughout.

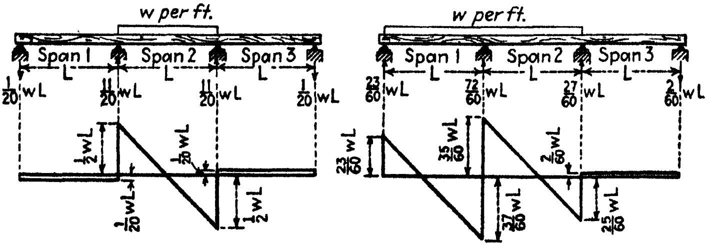


Curve of shears for uniform load = w per ft. on spans 1 and 3 only

Curve of shears for uniform load = w per ft. over entire structure

FIG. 14.—Continuous girder. Curves of shears for uniform load.

Note that it may simplify the computations of some trusses if instead of assuming a tension of unity in the redundant bars, a component of the stress is taken as unity, *e.g.*, the horizontal component. However, since this simplification must, in general, be used with much care, the theory is more easily understood in its derivation if the stress, instead of the component, in a redundant bar be taken as unity.



Curve of shears for uniform load = w per ft. on span 2 only

Curve of shears for uniform load = w per ft. on spans 1 and 2 only, This loading gives maximum shear in middle and end spans

FIG. 15.—Continuous girder. Curves of shears for uniform load.

$$\text{Now, } W = \text{total work} = \sum \frac{(S_1X + S_2Y + P)^2L}{2AE}$$

Hence, for minimum values,

$$\frac{\delta W}{\delta X} = \sum \frac{S_1X + S_2Y + P}{A} \times S_1L = 0 \qquad (19)^*$$

$$\frac{\delta W}{\delta Y} = \sum \frac{S_1X + S_2Y + P}{A} \times S_2L = 0 \qquad (20)^*$$

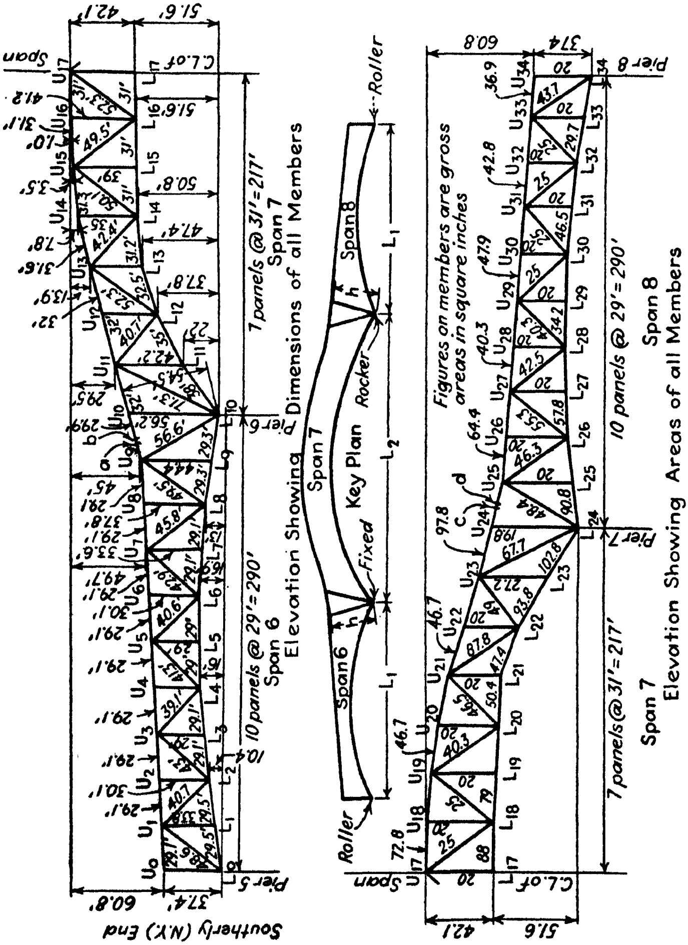


FIG. 16.—Trusses of Lake Champlain Bridge.

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*Equations (19) and (20) may also be obtained by the method of deflection as follows:

$$\delta_a + \delta_b \text{ for } U_9U_{10} = \sum \frac{(S_1X + S_2Y + P)S_1L}{A} = 0$$

$$\delta_a + \delta_b \text{ for } U_{24}U_{25} = \sum \frac{(S_1X + S_2Y + P)S_2L}{A} = 0$$

Now consider the condition for case where load is applied to one span only.

Load on Span 6 *Only.*—For this condition $P = 0$ for spans 7 and 8, the stresses in span 6 are expressed in terms of P and S_1 only, and the stresses in span 8 in terms of S_2 only; therefore, all terms in the above equations containing S_2 disappear for span 6; all terms containing P disappear for spans 7 and 8; and all terms containing S_1 disappear for span 8. We therefore obtain the following expressions in which the effect of each span is stated separately:

$$\frac{\delta W}{\delta X} = \overset{\text{Span 6}}{\sum \frac{(S_1X + P)S_1L}{A}} + \overset{\text{Span 7}}{\sum \frac{(S_1X + S_2Y)S_1L}{A}} + \overset{\text{Span 8}}{0} = 0 \qquad (21)$$

$$\frac{\delta W}{\delta Y} = \overset{\text{Span 7}}{\sum \frac{(S_1X + S_2Y)(S_2L)}{A}} + \overset{\text{Span 8}}{\sum \frac{S_2{}^2YL}{A}} = 0 \qquad (22$$

From Eq. (21):

$$\overset{\text{Span 6}}{\sum \frac{S_1{}^2LX}{A}} + \overset{\text{Span 6}}{\sum \frac{PS_1L}{A}} + \overset{\text{Span 7}}{\sum \frac{S_1{}^2LX}{A}} + \overset{\text{Span 7}}{\sum \frac{S_1S_2LY}{A}} = 0 \qquad (23)$$

and from Eq. (22):

$$\overset{\text{Span 7}}{\sum \frac{S_1S_2LX}{A}} + \overset{\text{Span 7}}{\sum \frac{S_2{}^2YL}{A}} + \overset{\text{Span 8}}{\sum \frac{S_2{}^2YL}{A}} = 0 \qquad (24)$$

Equations (23) and (24) may be written as follows:

$$aX + bY + m = 0$$

and

$$bX + cY = 0$$

in which

$$a = \sum \frac{S_1^2L}{A} \text{ for spans 6 and 7 combined}$$

$$b = \sum \frac{S_1S_2L}{A} \text{ for span 7}$$

$$c = \sum \frac{S_2^2L}{A} \text{ for spans 7 and 8 combined}$$

$$= a, \text{ since structure is symmetrical}$$

$$m = \sum \frac{PS_1L}{A} \text{ for span 6}$$

Hence

$$bX + aY = 0$$

Therefore

$$Y = -\frac{b}{a}X \tag{25}$$

Also,

$$aX - \frac{b^2}{a}X + m = 0$$

Whence

$$X = -\frac{am}{a^2 - b^2} \tag{26}$$

Note that a, b, and c are constants of the structure and not affected by position of load.

Load on Span 7 Only.—For this case P has a numerical value for span 7 but equals zero for spans 6 and 8; otherwise, conditions are as stated in previous case; hence for this case we obtain the following expressions:

$$\frac{\delta W}{\delta X} = \overset{\text{Span 6}}{\sum \frac{S_1^2XL}{A}} + \overset{\text{Span 7}}{\sum \frac{(S_1X + S_2Y + P)S_1L}{A}} + \overset{\text{Span 8}}{0} = 0 \tag{27}$$

$$\frac{\delta W}{\delta Y} = \overset{\text{Span 6}}{0} + \overset{\text{Span 7}}{\sum \frac{(S_1X + S_2Y + P)S_2L}{A}} + \overset{\text{Span 8}}{\sum \frac{S_2^2YL}{A}} = 0 \tag{28}$$

Therefore

$$\overset{\text{Span 6}}{\sum \frac{S_1^2XL}{A}} + \sum \frac{S_1^2LX}{A} + \overset{\text{Span 7}}{\sum \frac{S_1S_2YL}{A}} + \sum \frac{PS_1L}{A} = 0 \tag{29}$$

$$\overbrace{\sum\frac{S_1S_2XL}{A}+\sum\frac{S_2{}^2YL}{A}}^{\text{Span 7}}+\overbrace{\sum\frac{PS_2L}{A}+\sum\frac{S_2{}^2YL}{A}}^{\text{Span 8}}=0 \quad (30)$$

Equations (29) and (30) may be written

$$aX + bY + n = 0 \quad (31)$$
$$bX + aY + p = 0 \quad (32)$$

In preceding equations, a and b have same values as previously

$$n = \sum\frac{PS_1L}{A} \text{ for span 7}$$

$$p = \sum\frac{PS_2L}{A} \text{ for span 7}$$

Solving foregoing equations:

$$Y = \frac{ap - bn}{b^2 - a^2} \quad (33)$$

$$X = \frac{an - bp}{b^2 - a^2} \quad (34)$$

It follows that,

$$Y = \frac{ap - bn}{an - bp}X \quad (35)$$

In Eqs. (33) and (34) p and n are the only variables affecting the relation between the stresses in the redundant members for any given structure.

For load over entire length of span 7, with other spans unloaded, $n = p$ and $X = Y$, since structure is symmetrical. Hence for this condition,

$$X = Y = -\frac{p}{a + b}$$

Load on Span 8 *Only*.—Since the entire structure is symmetrical, the values of X and Y for a load on span 8 at any given distance from its right end are the same, respectively, as the values for Y and X for a load at an equal distance from the left end of span 6.

It should be observed that the introduction of an internal stress in any bar of a truss can cause no horizontal pier reactions, since, in order for such reactions to exist, a horizontal external force must be applied to the structure; otherwise, ΣH would not equal zero. Under such conditions horizontal equilibrium is maintained by the trusses themselves; *e.g.*, if a force of unity is assumed to replace bar U_9U_{10}, the truss in span 6 will be held in horizontal equilibrium by the truss in span 7 exerting a horizontal force at L_{10}.

A two-span continuous truss is indeterminate only to the first degree. The most convenient bar to select for the redundant is the chord bar, the center of moments of which lies on the vertical line through the intermediate reaction point. If X is the stress in this chord bar, and the previous notation is used,

$$S = S_1X + P$$

and

$$W = \text{total work} = \sum (S_1X + P)^2\frac{L}{2AE}$$

Hence, for minimum values, the following equation is obtained in place of Eq. (19) for a three-span continuous truss:

$$\frac{\partial W}{\partial X} = \sum (S_1X + P)S_1\frac{L}{A} = 0$$

This agrees as it should with the value obtained from Eq. (19) if Y is placed $= 0$. The remaining expressions in this article for a three-span continuous truss may be simplified in a similar manner to fit the case of the two-span structure.

12. Application of Theory to Continuous Trusses.—The application of the foregoing expressions to the determination of the live stresses in spans 6, 7, and 8 follows.

1. Compute S_1, S_2, L/A, S_1L/A, S_1S_2L/A, S_1^2L/A for each bar in structure, and get summations, thus obtaining constants of structure $a = \Sigma S_1^2L/A$ and $b = \Sigma S_1S_2L/A$.
2. Compute P for each bar for load of unity at each panel point with bars U_9U_{10} and $U_{24}U_{25}$ omitted (*i.e.*, with each span acting as an end-supported span); multiply by S_1L/A, and get summations, thus obtaining m, n, and p for load unity at each panel point.
3. Solve equations previously deduced for values of X and Y for load at each panel point, and from these determine actual

stress in each bar, by use of expression $S = P + S_1X + S_2Y$, thus obtaining influence data sufficient to determine method of loading and value of maximum stress of each character in each bar.

Note that the value of the various terms in span 8 equals those in span 6 and need not be computed.

A simple analytical method of computing the various bar stresses in trusses with inclined chords such as those shown in Fig. 16 follows.

Compute H.C. (horizontal component) of stress in every member of each of the spans due to upward force of unity at each end, *considering truss as fixed at other end.* The actual stress in every member due to load unity at any panel point or to stress unity in bars U_9U_{10}, etc., may then be found as follows for each of two cases:

Case A. For any bar to left of load, H.C. equals product of left reaction due to panel load of unity at loaded panel point or to stress unity in U_9U_{10} and H.C. due to upward force of unity at left end.

Case B. For any bar to right of load, H.C. due to panel load of unity at loaded panel point equals product of right reaction and H.C. due to upward force of unity at right end of span. For stress in U_9L_{10} due to S_1, note that H.C. of diagonal stress equals H.C. of bottom-chord stress as found under case *A* plus (algebraically) H.C. in U_9U_{10} corresponding to stress unity in that member.

TABLE I.—CHORD-BAR STRESSES DUE TO END REACTIONS OF UNITY—SPAN 6

Panel point	*M* at panel point due to force unity acting up. Truss assumed to be fixed at L_{10} for force at L_0, and at L_0 for force at L_{10}. (No stress in latter case in bar U_9U_{10})		Chord-bar stresses					
	Unity at L_0	Unity at L_{10}	Bar	Lever arm for H.C. of bar stress	H.C. stress = *M* ÷ lever arm, unity acting up		Actual stress, unity acting up	
					At L_0	At L_{10}	At L_0	At L_{10}

Tabular forms should be prepared before making computations; suggested headings of such forms are given in Tables I to IX inclusive:

In all computations, obtain results to three decimal places, using Thatcher slide rule or computing machine.

$$\text{In all cases}\begin{cases}\text{M} = \text{moment at panel point.}\\ \text{H.C.} = \text{horizontal component of bar stress.}\\ \text{V.C.} = \text{vertical component of bar stress.}\end{cases}$$

Since diagonal stresses are obtained from differences between horizontal components in chords, it is especially important that chord-bar stresses should be carried out to at least three places of decimals.

TABLE II.—WEB STRESSES DUE TO END REACTIONS OF UNITY—SPAN 6

	Unity up at L_0				Unity up at L_{10}			
Bar	H.C. top-chord bar in panel	H.C. bottom-chord bar in panel	H.C. bar	Stress in bar	H.C. top-chord bar in panel	H.C. bottom-chord bar in panel	H.C. bar	Stress in bar

NOTE: H.C. diagonal stress in any panel equals algebraic sum of H.C. of chord stresses. Signs must be given very careful consideration in determining web stresses which should also be checked by the method of moments.

TABLE III.—STRESSES P DUE TO FORCE UNITY ACTING DOWN AT PANEL POINTS—SPAN 6
(Trusses assumed as end supported)

	Load at panel point									
Bar	1	2	3	4	5	6	7	8	9	Sum

The values in this table may be computed as previously explained.

TABLE IV.—BAR STRESSES S_1 DUE TO TENSION OF UNITY IN U_9U_{10}; ALSO VALUES OF S_1^2L/A—SPAN 6

Bar	Stress = S_1	L, feet	A, inches	L/A	S_1L/A	S_1^2L/A All values in this column are positive regardless of sign of S_1
Summation						

The values in this table may be computed as previously explained.

TABLE V.—CHORD-BAR STRESSES DUE TO END REACTION OF UNITY ACTING UP AT L_{10}—SPAN 7
(See also Table I)

Panel point	Moment at panel point due to force unity acting up at L_{10}. Truss assumed fixed at L_{24} (no stress in U_9U_{10})	Bar	Lever arm for H.C. of bar stress	H.C.	Stress

TABLE VI.—STRESSES IN DIAGONALS DUE TO END REACTION OF UNITY ACTING UP AT L_{10}—SPAN 7

Bar	H.C. top-chord bar in panel	H.C. bottom-chord bar in panel	H.C. bar	Stress in bar

Values in Table VIII may be computed as previously explained. In view of symmetry, computations need be made for bars in one-half of truss only, but it is probably advisable to set down the results for all bars. The stress in bar L_{23}-L_{24} with load at 23 is the same as stress in bar L_{10}-L_{11} with load at 11, and similarly with loads at other panel points; hence, if stresses in bar

TABLE VII.—STRESSES IN VERTICALS DUE TO END REACTIONS OF UNITY ACTING UP AT L_{10}—SPAN 7

H.C. chord bars meeting at panel points at that end of vertical at which no diagonals meet		Horizontal and vertical projections of chord bars meeting at panel point		V.C. stress in chord bars	Stress = algebraic sum of vertical components in chord bars meeting at panel point	
Bar	H.C.	Vertical	Horizontal		Bar	Stress

TABLE VIII.—STRESSES P DUE TO FORCE UNITY ACTING DOWN AT PANEL POINTS—SPAN 7
(Trusses assumed as end supported)

Bar	Load at panel point													
	11	12	13	14	15	16	17	18	19	20	21	22	23	Sum

L_{10}-L_{11} are computed for all panel points, the stresses in L_{23}-L_{24} may be written by reversing the stresses in L_{10}-L_{11}.

TABLE IX.—BAR STRESSES S_1 AND S_2 DUE TO UNIT TENSION IN U_9U_{10} AND $U_{24}U_{25}$, RESPECTIVELY; ALSO VALUES OF S_1^2L/A AND S_1S_2L/A—SPAN 7

Bar	S_1	S_2	L, feet	A, inches	L/A	S_1L/A	S_1^2L/A all values positive	S_2L/A	S_1S_2L/A values may be positive or negative depending upon signs of S_1 and S_2
Summations									

The values of S_1 and S_2* may be computed as previously explained. S_2 for bars in right half of span is same as S_1 for corresponding bars in left half of span. S_1 for bar U_{23}-$U_{24} = 0$. S_2 for bar U_{10}-$U_{11} = 0$.

---

* Bar stresses $S_1 + S_2$ should be checked by computing the combined value of $S_1 + S_2$ for each case.

TABLE X.—VALUES OF PS_1L/A FOR LOAD UNITY AT PANEL POINTS—SPAN 6

Bar	S_1L/A	PS_1L/A for load at panel point indicated below								
		1	2	3	4	5	6	7	8	9
Summations										

TABLE XI.—VALUES OF PS_1L/A FOR LOAD UNITY AT PANEL POINTS SPAN 7

Bar	S_1L/A	PS_1L/A for load at panel point indicated below												
		11	12	13	14	15	16	17	18	19	20	21	22	23
Summations														

NOTE: $\sum \frac{PS_1L}{A}$ for load at any panel point equals $\sum \frac{PS_2L}{A}$ for load at corresponding panel point on other half of truss.

Application of the foregoing method gives the following constants of structure for the Lake Champlain Bridge. Values of constants for other continuous bridges of the same type are given in the footnote on next page.[1]

$$a = \sum \frac{S_1^2L}{A}$$

Flanking spans........ 8.82

Centre spans.......... 12.83

Total............... 21.65

$$b = \sum \frac{S_1S_2L}{A} = \quad 5.73$$

$$a^2 - b^2 = \quad 435.89$$

$$\frac{a}{a^2 - b^2} = \frac{21.65}{435.89} \quad 0.0496$$

These constants combined with the values of $\sum \frac{PS_1L}{A}$ as determined in Tables X and XI may be substituted in Eqs. (25), (26), (34), and (35) to determine the values of X and Y for any particular loading.

Figure 17 shows an influence line for H.C. of stress in U_9U_{11}; also for vertical reaction at L_0. Corresponding values as computed by the three-moment equation, assuming trusses to have constant value of I, are also shown on the diagram for comparison.

13. Effect upon Stresses Due to Settlement of Supports—Lake Champlain Bridge.—An objection to the use of continuous trusses is that settlement of the piers affects the bar stresses. This objection does not exist when the piers are on bedrock; and its influence is not large for slight settlement. The effect of such settlement may be determined as follows:

Case A. Settlement of One End Support.—Let X and Y = assumed tension in U_9U_{10} and $U_{24}U_{25}$, respectively, due to settlement, Δ_0 ft., of the support at L_0.

Let bars U_9U_{10} and $U_{24}U_{25}$ be assumed to be cut at *a-b* and *c-d* as indicated in Fig. 16.

---

[1] Values of constants for other bridges of same type as Lake Champlain Bridge are given below:

CAPE COD BRIDGES

$a = \sum \frac{S_1^2L}{A}$	Flanking spans............	10.2492
	Centre spans..............	12.3642
	Total....................	22.6134
$b = \sum \frac{S_1S_2L}{A} =$		6.2450
$a^2 - b^2 =$		472.36
$\frac{a}{a^2 - b^2} = \frac{22.6134}{472.36} =$		0.04787

LITTLE BAY BRIDGE

$a = \sum \frac{S_1^2L}{A}$	Flanking span.............	7.297
	Centre span...............	9.905
	Total....................	17.202
$b = \sum \frac{S_1S_2L}{A} =$		4.949
$a^2 - b^2 =$		271.415
$\frac{a}{a^2 - b^2} = \frac{17.202}{271.415} =$		0.0634

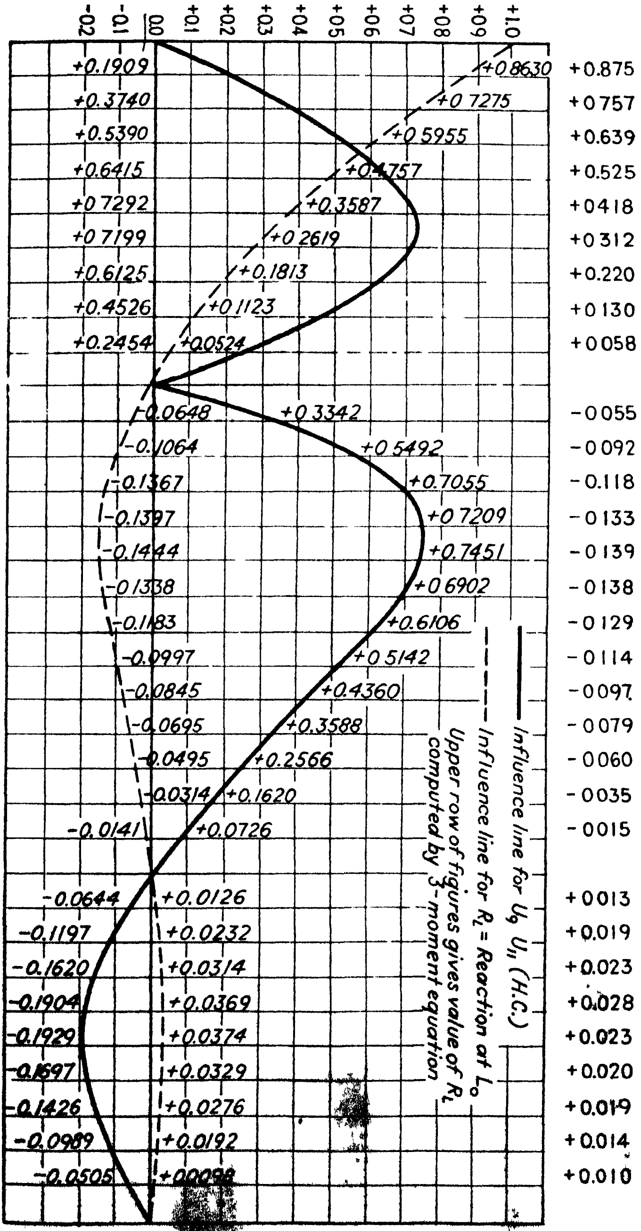

FIG. 17.—Influ[illegible]es for Lake Champlain Bri[illegible] trusses.

Let δ_a = deflection of point a in direction U_9U_{10}.
δ_b = deflection of point b in direction $U_{10}U_9$.
δ_c = deflection of point c in direction $U_{24}U_{25}$.
δ_d = deflection of point d in direction $U_{25}U_{24}$.
h = normal distance from L_{10} to U_9U_{10} = normal distance from L_{24} to $U_{24}U_{25}$.

Then[1]

$$\delta_a + \delta_b = 0 \qquad \text{and} \qquad \delta_c + \delta_d = 0$$

Now

$$\delta_a = -\frac{\Delta_0 h}{L_1} + \overset{\text{Span 6}}{\sum}\frac{XS_1{}^2L}{AE}$$

since the structure rotates about L_{10} and

$$\delta_b = \overset{\text{Span 7}}{\sum\frac{XS_1LS_1}{AE} + \sum\frac{YS_2LS_1}{AE}}$$

But, from Art. 11

$$\overset{\text{Span 6}}{\sum}\frac{S_1{}^2L}{A} + \overset{\text{Span 7}}{\sum}\frac{S_1{}^2L}{A} = a$$

and

$$\overset{\text{Span 7}}{\sum}\frac{S_2S_1L}{A} = b$$

Therefore

$$E(\delta_a + \delta_b) = -\frac{\Delta_0 hE}{L_1} + aX + bY = 0 \tag{36}$$

Similarly, for bar $U_{24}U_{25}$ we obtain:

$$\delta_c + \delta_d = \overset{\text{Span 7}}{\sum\frac{S_2S_1XL}{AE} + \sum\frac{S_2{}^2YL}{AE}} + \overset{\text{Span 8}}{\sum}\frac{S_2{}^2YL}{AE} = 0$$

$$E(\delta_c + \delta_d) = bX + aY = 0 \tag{37}$$

Therefore

$$Y = -\frac{bX}{a} \qquad \text{See also Eq. (25)} \tag{38}$$

---

[1] See Theory of Structures, 3d ed., 3d impression, Art. 197, Spofford.

which when inserted in Eq. (36) gives:

$$-\frac{\Delta_0 hE}{L_1} + aX - \frac{b^2X}{a} = 0$$

Therefore

$$X = \frac{\Delta_0 hE}{L_1\left(a - \frac{b^2}{a}\right)} = \frac{a\Delta_0 hE}{L_1(a^2 - b^2)} \tag{39}$$

In Eq. (39) X is positive when Δ_0 is downward.

Case B. Settlement of One Intermediate Support, Lake Champlain Bridge.—Let Δ_{10} = settlement in feet of intermediate support at L_{10}. The effect of this settlement is the same as if all other supports were raised the distance Δ_{10}, in which case span 6 and span 7 would rotate about L_{10}.

Then

Span 6

$$\delta_a = \frac{\Delta_{10}h}{L_1} + \sum\frac{S_1{}^2LX}{AE}$$

Span 7

$$\delta_b = \frac{\Delta_{10}h}{L_2} + \sum\frac{S_1{}^2LX}{AE} + \sum\frac{S_1LS_2Y}{AE}$$

Therefore

$$E(\delta_a + \delta_b) = 0 = \frac{\Delta_{10}hE}{L_1} + \frac{\Delta_{10}hE}{L_2} + aX + bY$$

Therefore

$$aX + bY = -\Delta_{10}E\left(\frac{h}{L_1} + \frac{h}{L_2}\right) \tag{40}$$

Also,

Span 7

$$\delta_c = -\frac{\Delta_{10}h}{L_2} + \sum\frac{S_1S_2LX}{AE} + \sum\frac{S_2{}^2LY}{AE}$$

Span 8

$$\delta_d = \sum\frac{S_2{}^2LY}{AE}$$

$E(\delta_c + \delta_d) = 0 =$ [illegible] $+ bX + aY$

hence

$$bX + aY = \frac{\Delta_{10}hE}{L_2} \qquad \text{and} \qquad Y = -\frac{bX}{a} + \frac{\Delta_{10}hE}{aL_2} \tag{41}$$

But from Eq. (40),

$$aX + bY = -\Delta_{10}E\left(\frac{h}{L_1} + \frac{h}{L_2}\right) \tag{42}$$

Therefore

$$X(b^2 - a^2) = \Delta_{10}E\left(\frac{bh}{L_2} + \frac{ah}{L_1} + \frac{ah}{L_2}\right) \tag{43}$$

and

$$X = -\Delta_{10}hE\left[\frac{a}{L_1(a^2 - b^2)} + \frac{1}{L_2(a - b)}\right] \tag{44}$$

The method of determining X and Y given in Appendix A may be used in checking the foregoing equations.

14. Numerical Illustration of Effect of Settlement—Lake Champlain Bridge.—Values of X and Y for settlement of L_0 of 1 in. are found from Eqs. (38) and (39) and from constants of Art. 11 to be:

$$X = \frac{54.5}{12 \times 290}0.0496E = +0.0007767E$$

$$Y = -0.0007767E \times \frac{5.73}{21.65} = -0.0002056E$$

For simultaneous settlement of 1″ at L_0 and L_{34},

$$X = Y = 0.00057E$$

Since in the design the values of S_1 and S_2 for all members have been obtained, the effect upon all members of the foregoing changes in the stresses in the redundant members can be readily obtained.

The changes in end reactions corresponding to foregoing changes in the redundant members are as follows:

For settlement of 1 in. at L_0,

$$\text{Decrease in } R_0 = \frac{0.0007767E \times 54.5}{290} = 0.0001459E$$

$$\text{Increase in } R_{34} = 0.0002056E \times \frac{54.5}{290} = 0.0000387E$$

For simultaneous settlement of 1 in. at L_0 and L_{34}, decrease in reactions at L_0 and at L_{34}

$$= (0.0001459 - 0.00003868)E = 0.0001072E = (0.0005711E) \times 54.5/290$$

Values of X and Y for settlement of L_{10} of 1 in. may be found from Eqs. (44) and (41).
From Eq. (44),

$$X = -\left[X \text{ due to settlement of 1 in. at } L_0 + \frac{E \times 54.5}{12(434)(15.92)}\right]$$
$$= -(0.0007767 + 0.0006573)E = -0.001434E$$

From Eq. (41),

$$Y = -\frac{b}{a}x + \frac{\Delta_{10}hE}{aL_2}$$
$$= -\frac{5.73}{21.65}(-0.001434E) + (0.0006573E) \times \frac{15.92}{21.65}$$
$$= (0.0082168 + 0.0104642)\frac{E}{21.65}$$
$$= +0.0008629E$$

The changes in reactions and bar stresses can be readily computed from the values of X and Y just determined.

For simultaneous settlements of 1 in. at L_{10} and L_{24}

$$X = (-0.001434 + 0.0008629)E = -0.0005711E = Y$$

Since the simultaneous settlements of 1 in. of L_{10} and L_{24} have same numerical effect as simultaneous raising of L_0 and L_{34}, the preceding values of X and Y should give the values previously found for above conditions with signs reversed, which is seen to be the case.

The maximum value of X and Y for which bars U_9U_{10} and $U_{24}U_{25}$ were designed was +1,237,000 lb.; hence, the effect upon these members of a differential settlement of 1 in. at either L_0 or L_{10} is but a small percentage of the design stress. The effect of the settlement is to increase the maximum stresses in some bars and diminish them in others, and in some bars the percentage effect may be greater than in the bars considered.

In determining the effect of pier settlement it should be borne in mind that for heavy riveted trusses the deflections are usually considerably less than those found theoretically, using for E the ordinary modulus of elasticity. Gusset plates, lateral and floor systems, etc., all have the same effect as increasing E; but even if E is increased by 50 per cent of its value for steel bars, the effect of pier settlement is still seen to be relatively small in this bridge.

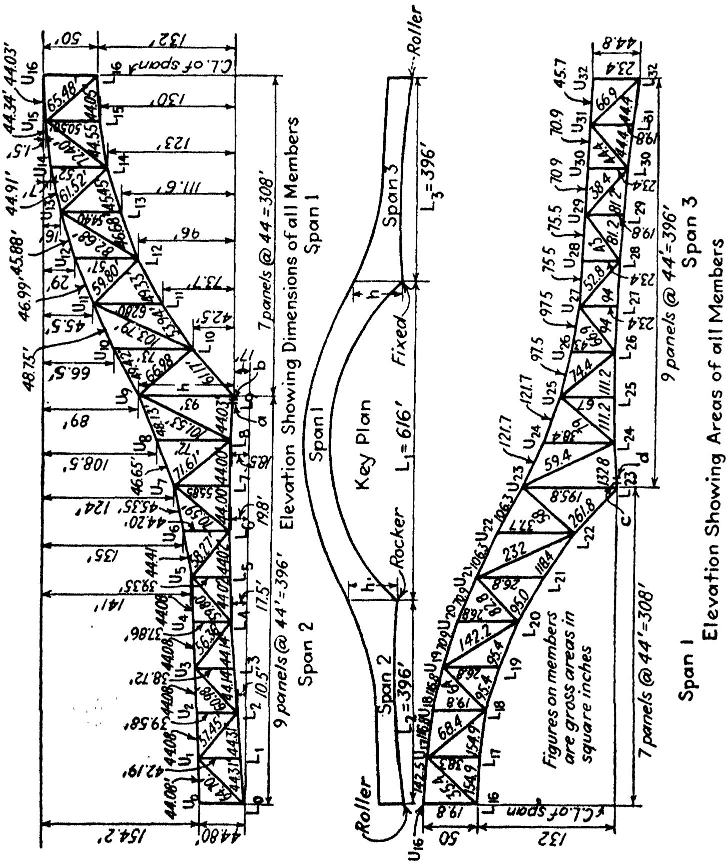

FIG. 18.—Trusses of Cape Cod Canal Bridges.

15. Effect upon Stresses Due to Settlement of Supports—Cape Cod Bridges.—In the computation of these bridges, the unknowns X and Y were for convenience in figuring taken as the H.C. of stresses in L_9L_{10} and $L_{22}L_{23}$, which equal the H.C. of stresses in L_8L_9 and $L_{23}L_{24}$, respectively; and S_1 and S_2 were taken

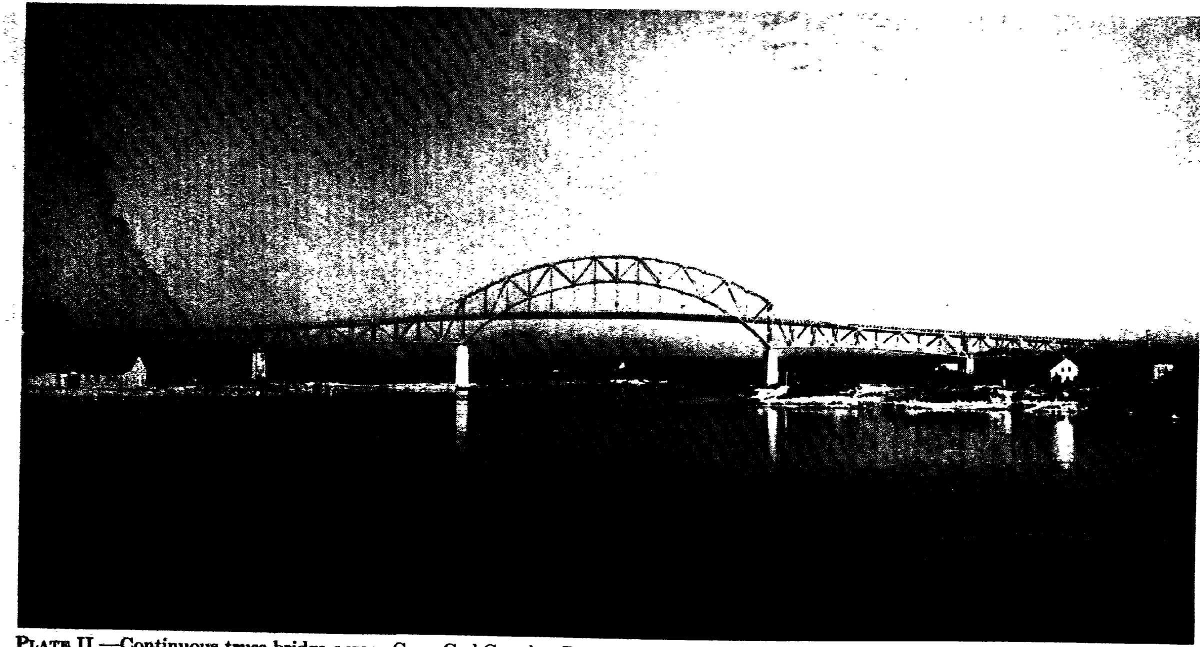

PLATE II.—Continuous truss bridge across Cape Cod Canal at Bourne, Mass., built for the United States War Department in 1933–1935.

(Facing page 42.)

as the bar stresses due to a horizontal component of unity in each of the foregoing bars.

Case A. Settlement of One End Support.—Referring to Fig. 18 and assuming bars L_8L_9 and $L_{23}L_{24}$ cut as shown at *ab* and *cd*, we get the following equations:

Let δ_a = H.C. of deflection of point a in direction L_8L_9.
δ_b = H.C. of deflection of point b in direction L_9L_8.
δ_c = H.C. of deflection of point c in direction $L_{23}L_{24}$.
δ_d = H.C. of deflection of point d in direction $L_{24}L_{23}$.

For this structure $a = c = \sum \frac{S_1^2L}{A}$ for span 1 + $\sum \frac{S_1^2L}{A}$ for span 2, and $b = \sum \frac{S_1S_2L}{A}$ for span 1.

Owing to settlement Δ_0 ft. at L_0 we obtain the following expression:

$$\delta_a = +\frac{\Delta_0 h}{L_2} + \overset{\text{Span 2}}{\sum} \frac{S_1^2XL}{AE}$$

$$\delta_b = \overset{\text{Span 1}}{\sum \frac{S_1^2XL}{AE} + \sum \frac{S_1S_2LY}{AE}}$$

Therefore

$$E(\delta_a + \delta_b) = \frac{\Delta_0 hE}{L_2} + aX + bY = 0 \qquad (45)$$

also

$$\delta_c = \overset{\text{Span 1}}{\sum \frac{S_2^2YL}{AE} + \sum \frac{S_2S_1XL}{AE}}$$

$$\delta_d = \overset{\text{Span 3}}{\sum} \frac{S_2^2YL}{AE}$$

Therefore

$$\delta_c + \delta_d = bX + aY = 0 \qquad (46)$$

Therefore

$$Y = -\frac{b}{a}X \qquad \text{See also Eq. (25)}$$

which when inserted in equation (45) gives the following result:

$$X = -\frac{a\Delta_0 hE}{L_2(a^2 - b^2)} \tag{47}$$

Note that the foregoing is identical with Eq. (39) except for the sign and change in nomenclature of span length.

Case B. Settlement of One Intermediate Support.—Let Δ_9 = settlement of L_9.

1. Assume L_0 and all other points in span 2 to be temporarily lowered the distance Δ_9.

Then

Span 1

$$\delta_b = -\Delta_9\frac{h^*}{L_1} + \sum\frac{XS_1^2L}{AE} + \sum\frac{YS_2LS_1}{AE}$$

2. Determine δ_a due to raising L_0 the distance Δ_9.

Span 2

$$\delta_a = -\frac{\Delta_9 h}{L_2} + \sum\frac{S_1^2LX}{AE}$$

It follows that

$$E(\delta_a + \delta_b) = 0 = -\Delta_9 E\left(\frac{h}{L_1} + \frac{h}{L_2}\right) + aX + bY \tag{48}$$

Also

Span 1

$$\delta_c = +\frac{\Delta_9 h}{L_1} + \sum\frac{XS_1S_2L}{AE} + \sum\frac{YS_2^2L}{AE}$$

Span 3

$$\delta_d = \sum\frac{YS_2^2L}{AE}$$

Therefore

$$E(\delta_c + \delta_d) = 0 = \frac{E\Delta_9 h}{L_1} + bX + aY \tag{49}$$

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* Under assumed condition, span 1 rotates about U_{23}, since bar $L_{23}L_{24}$ is assumed to be cut; hence, horizontal movement of L_9 equals horizontal movement of L_{23} equals $\Delta_9 \times 93/L_1$, hence $\delta_b = \Delta_9/L_1 \times 93$.

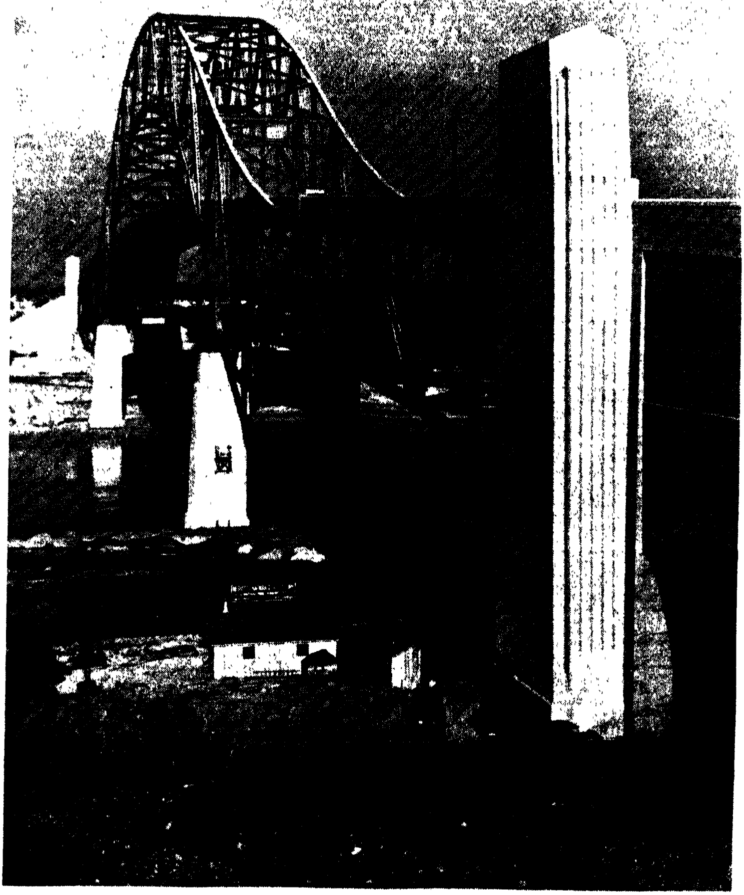

PLATE III.—Continuous truss bridge across Cape Cod Canal at Sagamore, Mass., built for the United States War Department in 1933–1935.

Solving Eqs. (48) and (49) gives

$$X = +\Delta_9 hE\left[\frac{a}{L_2(a^2 - b^2)} + \frac{1}{L_1(a - b)}\right] \tag{50}$$

$$Y = -\frac{b}{a}X - \frac{\Delta_9 hE}{aL_1} \tag{51}$$

[Note that Eqs. (50) and (51) are similar to Eqs. (44) and (41) except for signs and constants.]

16. Numerical Illustration of Effect of Settlement—Cape Cod Bridges.[1]—Let L_0 settle 1 in. = $\frac{1}{12}$ ft.
Then

$$X = -\frac{93}{12} \times E \times \frac{0.04787}{396} = -0.0009368E$$

and

$$Y = -\frac{b}{a}X = 0.0002587E$$

For simultaneous settlement of 1 in. at L_0 and L_{32}

$$X = (-0.0009368 + 0.0002587)E = -0.0006781E$$
$$= Y$$

For the foregoing case,

$$R_0 = R_{32} = \frac{(0.0006781E) \times 93}{396} = 0.0001593E$$

Let L_9 settle 1 in.
Then

$$X = \frac{1}{12} \times 93 \times E\left[\frac{0.04787}{396} + \frac{1}{616 \times 16.368}\right] \text{lb.}$$
$$= 0.0017055E$$

$$Y = -\left[\frac{6.245}{22.613}(0.0017055E) + \frac{93E}{12 \times 22.613 \times 616}\right]$$
$$= -0.0010274E$$

Evidently, for $\Delta_{23} = 1$ in.

$$X = -0.0010274E \quad \text{and} \quad Y = \quad 0.0017055E$$

---

[1] See Art. 12 for constants.

Hence, if Δ_9 and Δ_{23} each equal 1 in.,

$$X = Y = 0.0017055E - 0.0010274E = 0.0006781E$$

which agrees numerically with value for case of simultaneous settlement of both end supports as previously determined. (Sign is of opposite character, as should be the case.)

Note that the foregoing computations were carried out with great precision in order to verify results obtained by a method that requires subtraction of numbers of several decimal places. They are, of course, too precise for practical computations.

17. Erection of Continuous Trusses.—The erection of the continuous structures shown in Figs. 16 and 18 was accomplished by constructing each of the end arms on falsework, then cantilevering out the halves of the centre span and finally moving the two complete half structures by jacks until the trusses were connected.

The trusses were cambered so that the members in each span had their correct geometric lengths under full dead load and a specified uniform live load per foot over the entire span.[1]

In carrying out the provisions of the specifications, for the Lake Champlain Bridge shown in Fig. 16, span 8 was erected with shoes on pier 7, 3¼ in. forward (toward centre of span 7) of their dead-load positions, *i.e.*, (3¼ + 0.99) forward of centre line of pier, and with shoes on piers 5 and 8, 20 in. lower than normal, this latter provision being sufficient of itself to cause the following separation between the chords of the projecting ends:

	Inches
Top chord	12.92
Bottom chord	7.12

As a result of computations, the following figures were determined by the contractors, The American Bridge Company of New York:

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[1] Experience with other bridges indicates that for continuous bridges of larger span than the Lake Champlain Bridge it would be much simpler and probably equally satisfactory theoretically to camber for dead-load stresses only.

Shortening of span 7 due to camber equals horizontal deflection of L_{24} to left due to change in lengths of members for cambering purposes = 2.58 in.

Horizontal movement of L_{24} due to full dead load = 1.59 in. to right.

It follows that the length of span 7 at normal temperature under full dead load would equal 434 ft. minus 2.58 in. plus 1.59 in. or 434 ft. minus 0.99 in.; hence, final position of L_{24} with structure completed and carrying dead load should be 0.99 in. (say, 1 in.) forward of pier centre in order that it may be at pier centre under assumed live load of 500 lb. per foot.

If shoes on pier 7 should be erected in their dead-load positions, *i.e.*, 0.99 in. forward of centre of pier, the opening between projecting ends of span 7, if these are assumed to be supported on falsework so that members are unstressed, would equal

$$2.58 \text{ in.} - 0.99 \text{ in.} = 1.59 \text{ in.}$$

Horizontal deflections of U_{17} and L_{17} due to erection stresses with and without traveler for each half of arm were found to be as follows:

Deflection toward centre of span due to maximum erection stresses (traveler at projecting end):

U_{17}	4.63 in.
L_{17}	1.66 in.

Deflection toward centre of span due to erection stresses without traveler:

U_{17}	3.16 in.
L_{17}	1.04 in.

Change in total length of span 7 due to temperature change of 30° = 1.01 in.

In the actual erection, L_{24} was placed 3¼ in. forward of its dead-load position; hence, the opening between the projecting ends of span 7 would be as shown in following table, for case of normal temperature and correct bar lengths with traveler at end of each arm; also without traveler.

The tabular figures show that to close the span, provided the travelers are removed, the north half should be moved horizontally 3.38 in., closing bottom chord; then L_0 and L_{34} should be raised 2.41 in. each, closing top chord.

NOTE: $2.41/20 \times 12.92 = 1.56$ in.

		Inches
With traveler:[1]		
Top-chord opening	$12.92 + 1.59 - 3.25 - 9.26 =$	2.00
Bottom-chord opening	$7.12 + 1.59 - 3.25 - 3.32 =$	2.14
Without traveler:		
Top-chord opening	$12.92 + 1.59 - 3.25 - 6.32 =$	4.94
Bottom-chord opening	$7.12 + 1.59 - 3.25 - 2.08 =$	3.38

If temperature at time of closing is 30° above normal, and lengths overrun such that total length of span 7 is 1 in. more than figured length, top-chord opening will be

$$4.94 - 1.00 - 1.01 = 2.93 \text{ in.}$$

and that at bottom chord will be $3.38 - 2.01 = 1.37$ in.

To close bottom chord under these conditions, end would have to be raised 3.88 in., which would reduce opening at top chord to 0.42 in., and permit chord to be closed by raising ends 0.65 in. additional.

NOTE: $3.88/20.00 \times 7.12 = 1.37$ in.
$3.88/20.00 \times 12.92 = 2.51$ in.
$0.65/20.00 \times 12.92 = 0.42$ in.

If temperature is 30° below normal, and lengths underrun 1 in., the top-chord opening will be

$$4.94 + 2.01 = 6.95 \text{ in.}$$

and the bottom-chord opening

$$3.38 + 2.01 = 5.39 \text{ in.}$$

To close, right half of structure would have to be moved forward 3.75 in. leaving clearance of 3.20 in. at top and 1.64 in. at bottom. Under these conditions ends would have to be raised 4.58 in. to close bottom and 0.36 in. farther to close top chord.

NOTE: $4.58/20.00 \times 7.12 = 1.64$ in.
$4.58/20.00 \times 12.92 = 2.96$
$0.36/20.00 \times 12.92 = 0.24$

The extreme position of shoe on pier 7 would occur in preceding case and would equal $3.25 + 3.75 = 7.00$ in. forward of its dead-load position. The extreme position in the opposite direction, *i.e.*, back of its dead-load position, equals approximately 2 in. for temperature change of 60° plus 1 in. due to live-load stresses (1 in. equals increase in length due to 500 lb. per foot per truss); hence, total movement would equal

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[1] Traveler was not centred between trusses, and following figures are for the more heavily loaded truss:

$$7 \text{ in.} + 3 \text{ in.} = 10 \text{ in.}$$

therefore, rockers were set vertical when in mid-position or 2 in. forward of dead-load position.

After trusses of span 7 were connected, the truss ends over piers 6 and 8 were raised by calibrated hydraulic jacks to give the required net dead reactions consistent with truss loads at time jacking was done; *i.e.*, with steelwork in place and also with temporary tracks, forms for concrete floor, loading platforms, etc.

The following figures illustrate the necessary computations:

	Pounds
Net reaction due to trusses only	71,000
End panel load, west truss	29,000
Steel forms, track and end working platforms	15,000
Total end uplift to be applied by jacks at each end of west truss	115,000
End panel loads, east truss	29,000
Steel forms, track and loading platform	12,000
Total end uplift to be applied by jacks at each end of east truss	112,000

18. Specifications for Erection of Continuous Trusses.—The necessary factors to be considered in providing for erection are shown by the following extracts from the specifications for the Bourne and Sagamore Bridges across the Cape Cod Canal.

The trusses in all spans except 1 of each bridge are designed on the assumption that they will be completely supported on falsework during erection. The trusses in span 2 and 3 are designed to permit cantilever erection of the trusses in span 1, each half of which is designed to be erected as a cantilever without intermediate supports.

The erection stresses given on the stress sheets for spans 1, 2, and 3 are computed for the weight of the steel structure itself, together with a lateral force of 30 lb. per square foot on 1½ times the area of the structure as seen in elevation, and in addition a traveler of such weight and proportions that with its runway it will produce stresses in the various members equivalent to those which would be caused by a vertical load of 50 tons and a horizontal force of 5 tons acting at an elevation of 10 ft. above the center line of the top chord, both of these latter forces being assumed as applied at or directly above any panel point of either truss except the panel point at the center of span 1.

No falsework is to be used in span 1 except as authorized by the contracting officer, and span 1 shall not be obstructed in any way to interfere with navigation requirements at any time during the entire period of construction of the superstructures. If the contractor elects to employ a traveler causing greater stress in any of the members than that due to the traveler previously described or to make other changes in the method of erection which will increase the stress in any member over that shown in the contract plans, he shall be allowed no payment for any increase in the weight of steel thereby caused, which increase will be considered as required for erection purposes and covered by the unit prices of the contract.

Each truss in each cantilever arm of span 1 shall be erected and completely riveted at all panel points except the center before the two cantilever arms are riveted together. The steel members in the lateral bracing of all spans and the steel members in the floor system of spans 2 and 3 shall be erected simultaneously with the trusses. All stringers in spans 2 and 3 shall be erected in place before the cantilever arms are erected. The floor system in span 1 shall not be erected until the trusses of span 1 are completely riveted.

During the process of erection, rollers, shoes, and bearing plates on piers 3 and 4 of the Bourne Bridge and the abutments of the Sagamore Bridge for the trusses in spans 2 and 3 need not be in position, the trusses being supported on the jacking girders which are located over these piers and abutments, and bearing plates shall be provided to distribute properly the reactions from these jacking girders during the erection. The sole plates on the bottoms of the jacking girders and the said bearing plates shall be planed on their adjoining surfaces to permit longitudinal movement during erection. Any uplift which may be developed by the method of erection employed by the contractor shall be resisted by counterweights or by such other means as may be approved by the engineer so that the total resistance to uplift at any pier shall equal at least twice the computed uplift and more if required by the engineer.

After each cantilever arm in span 1 is completely riveted, but before the arms are riveted together, one of the side spans with its connected cantilever shall be moved longitudinally, and the ends of the trusses of spans 2 and 3 over piers 3 and 4 of the Bourne Bridge and over the abutments of the Sagamore Bridge shall be raised by hydraulic jacks to close the cantilever arms. The holes for all rivets required to connect the various members meeting at the center panel points of span 1 shall

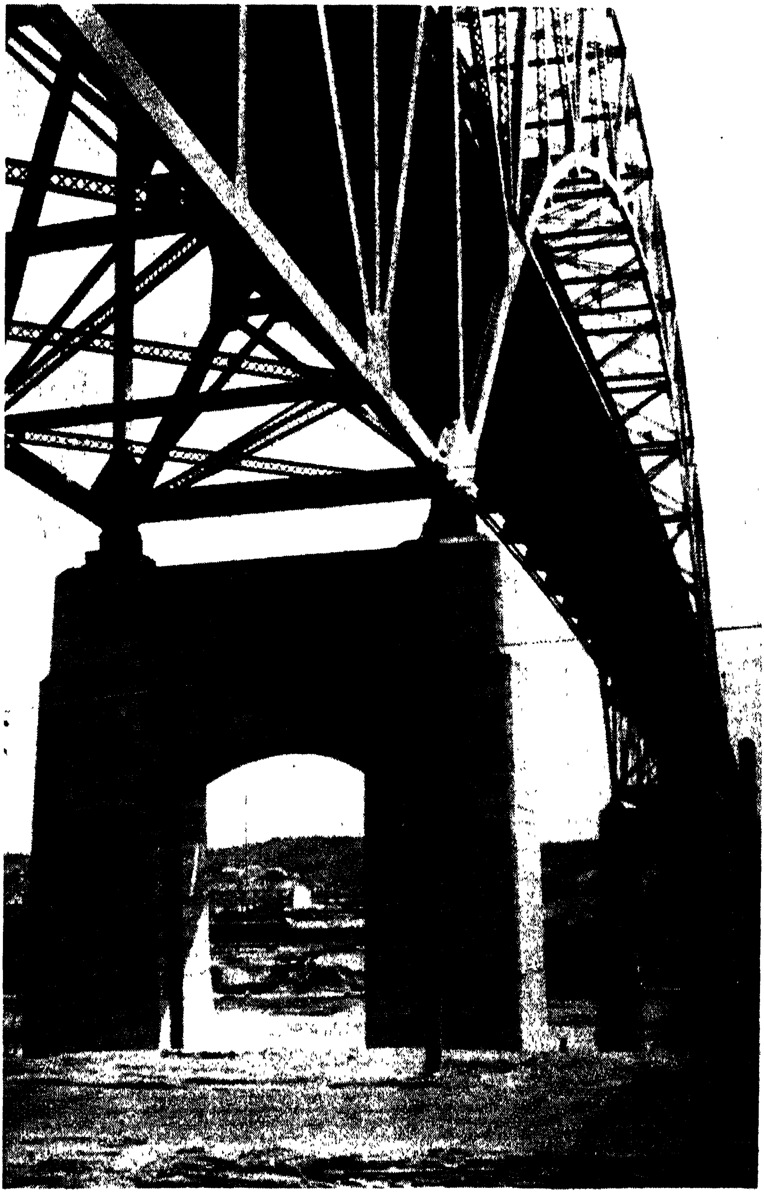

PLATE IV.—Continuous truss bridge across Cape Cod Canal at Bourne, Mass., built for the United States War Department in 1933–1935.

then be drilled to size in the field, and the rivets connecting these members shall be driven. The rivet holes shall be drilled, and the rivets driven only under weather conditions approved by the engineer.

After the completion of the riveting, the shoreward ends of spans 2 and 3 shall be raised by jacks to permit the insertion of shoes, roller nests, bearing plates, and shims, as necessary, and to give the required end reactions. The required magnitude of these end reactions shall be determined by the contractor after the completion of the shop drawings and the corresponding determination of the weight of steel in the superstructures and submitted to the engineer for approval and shall be such as to make the total end reaction of the completed structure at each shoreward end of each of the trusses in spans 2 and 3 due to the total dead weight of the structure including the concrete deck, railings, and other details equal to that required to give the dead stresses shown on the stress sheets in members $L_9 L_{10}$ and $L_{22} L_{23}$.

Careful records shall be made at each jack throughout the raising of the ends of each truss to determine the ratio between distance raised in inches and pressure applied in pounds, and from the values thus obtained the distance each truss end must be raised to give the correct reaction shall be determined. Devices satisfactory to the engineer and to such a degree of precision that the reactions applied to the trusses may be measured with an error not greater than 1 per cent shall be used in the jacking process.[1]

Care shall be taken in locating the position of the rocker nests on pier 1 of the Sagamore Bridge and pier 2 of the Bourne Bridge and the roller nests on piers 3 and 4 of the Bourne Bridge and the abutments of the Sagamore Bridge so that when the superstructures of spans 1, 2, and 3 are completed, including the concrete deck, railings, and other details, the center of each nest shall at a mean temperature of 50°F. lie in the vertical line passing through the center of the truss pin and the center of its respective base plate. The rocker nests shall be so placed that the rockers will be vertical at the average position of their extreme movement. If, after the ends of the trusses supported on piers 3 and 4 of the Bourne Bridge and on the abutments of the Sagamore Bridge are in their final positions on their bearings, the tops of the stringers and fascia girders adjoining either expansion joint are found to be at different levels, fillers shall be placed under the steel of the expansion-joint supports as directed by the engineer to correct such difference in level. Jacking girders are provided at the ends of spans 4 and 5 adjoining spans 2 and 3 of the Bourne Bridge in order that the ends of these spans may be adjusted in height, if necessary, to conform to the final elevations of the ends of spans 2 and 3.

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[1] Proving rings were used for this purpose. See reference to article on proving rings at end of chapter.

The foregoing specifications were slightly modified during the carrying out of the work as follows:

For convenience in erection, a portion of the floor system in span 1 was erected before the trusses in that span were riveted together, since it was found that this could be done without increasing the stresses in the various members beyond the allowable erection stress. Temporary roller nests were used under the jacking girders instead of planed bearing plates. The contractor was permitted on his own responsibility and with satisfactory results to use shop-rivet holes in place of field-drilled holes at connections of cantilever arms.

To facilitate erection, panel points 0 of spans 2 and 3 (see Fig. 18) were set originally 24½ and 32¼ in., respectively, below their normal positions, provision for this having been made in the piers by providing pockets of ample size. L_0 of span 2 was also set 17¼ in. forward of its normal position. After the steel in the trusses was erected, span 2 was jacked longitudinally until the bottom chords of the projecting ends of span 1 met at L_{16}. The L_0 ends of both spans 2 and 3 were then jacked vertically until the top chords met at L_{16}. The diagonals and laterals at the centre of span 1 which had been temporarily omitted were then inserted, and the riveting of span 1 completed. The trusses were then jacked up over piers 3 and 4 with 500-ton jacks, the pockets in the piers under the shoes filled with quick-setting concrete, and the shoes placed on shims at their approximate final elevation, following which their final elevation was determined by the use of proving rings.

In view of the amount of longitudinal movement, temporary roller nests were used under the jacks and the jacking girder; and to prevent accidents the jacking girders were kept blocked up on the masonry so that tipping of jacks or proving rings would not permit a sudden dropping of the bridge ends; the trusses were also guyed to prevent undue longitudinal movement.

19. Deflections of Continuous Trusses.—It should be observed that the actual deflections of riveted trusses are considerably smaller than the theoretical deflections if the latter are computed in the usual manner, *i.e.*, by using the gross areas of members and the value of E for steel bars in computing the deflection. Investigations of the Cape Cod Bridges by W. F. Hiltner in connection with his thesis for the Master's degree at the Massachusetts Institute of Technology in 1935 indicated that the effect of gusset plates, lattice bars, and batten plates on the main members was to increase the gross area applicable to the deflection by 9 per cent. In addition, the participation action of the laterals and the floor system increased the stiffness by 3 per cent.

The stiffness as actually determined in the field was considerably greater than the theoretical value as determined by using areas of main members only, even after allowing for the effects of gusset plates, etc., as given above, this excess being perhaps due in part to approximations in the value of *E* and to overrun of structural members, also to rigidity of joints. It should be observed, however, that so far as stress computations are concerned, the increased stiffness due to such causes, if considered to be distributed uniformly throughout the span, has the same effect as assuming a higher value for *E*. Such an assumption has no influence on the computation of stresses, since *E* cancels out of all equations, but does increase the effect upon the stresses due to settlement of piers.

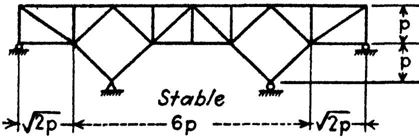

FIG. 19.—Statically determinate continuous structures.

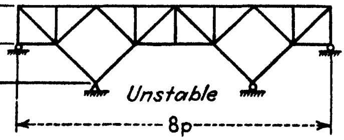

FIG. 20.—Unstable continuous truss.

20. Statically Determinate Continuous Trusses. General.—It is possible to build continuous trusses that are statically determinate, and examples of such trusses are shown by Figs. 19, 21, and 23. Such trusses were illustrated on pages 74 and 77 of "Die Graphische Statik der Baukonstruktionen" by Müller-Breslau, Volume 2, 1903, and discussed in the accompanying text; and were also illustrated by Figs. 512, 542, and 543 of Vol. 1 of the 1912 edition of the same book and discussed at considerable length on pages 515 *et seq.* Recently E. M. Wichert of Pittsburgh secured a patent for trusses of this type. The book entitled The Wichert Truss, by Dr. D. B. Steinman, listed in the references at end of the chapter, deals comprehensively with this type of truss.

So far as the writer knows, no trusses of this type have yet been completed in the United States, but the Pittsburgh-Homestead

Bridge across the Monongahela River, now under construction and designed by George S. Richardson, bridge engineer, Allegheny County Authority, Pittsburgh, Pa., includes such trusses.

Trusses of the type shown in foregoing figures are not made up of triangles, hence may not be statically determinate even if twice the number of joints equals the number of bars plus the number of reactions; *e.g.*, the truss in Fig. 20 is unstable owing to its proportions; such trusses should, therefore, never be used without a thorough investigation of their stability. This may be done by the application of the method of determinants given in "Die Graphische Statik der Baukonstruktionen," by Müller-

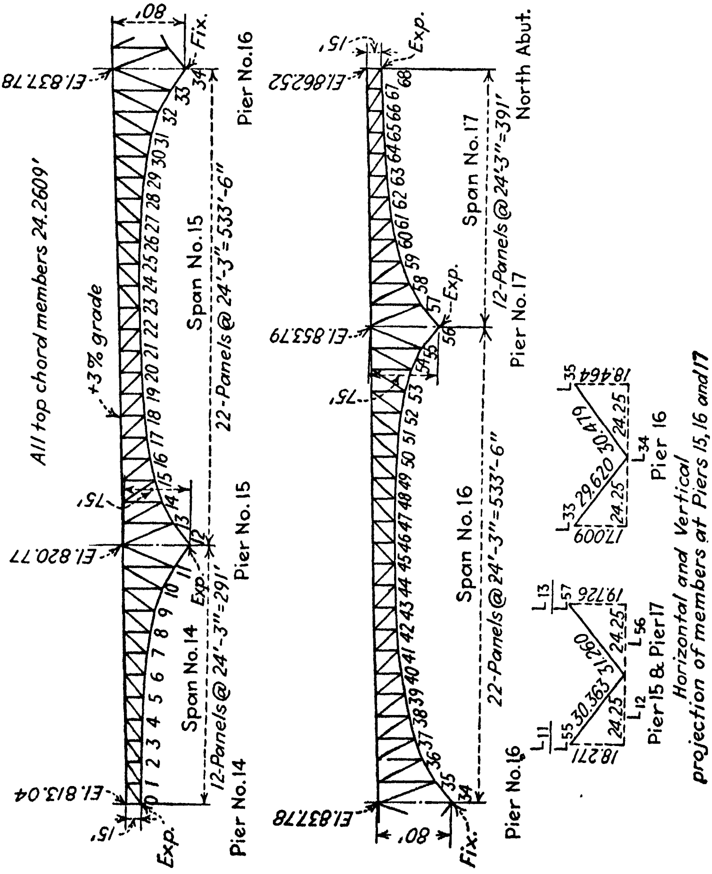

FIG. 21.—Trusses of the Pittsburgh-Homestead Bridge.

Breslau, pages 492 *et seq.*, and outlined by Waidelich in the article given in references at the end of the chapter; or else its stability should be tested by the construction of a simple model.

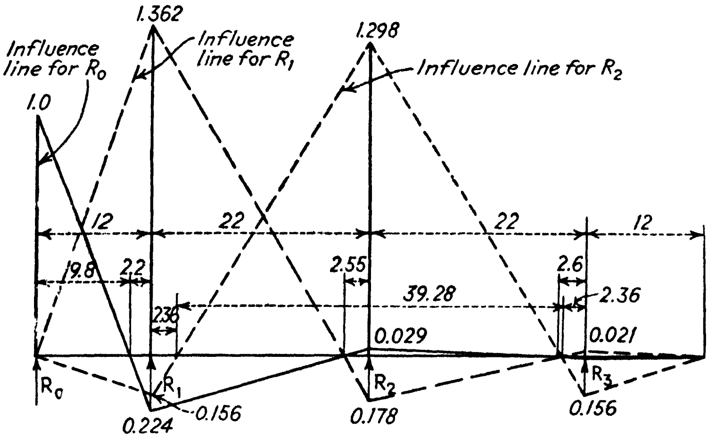

FIG. 22.—Influence lines for the Pittsburgh-Homestead Bridge.

21. Statically Determinate Continuous Trusses. Theory.—The truss illustrated by Fig. 21 is taken from the plans for the Pittsburgh-Homestead Bridge. It will be observed that the unknowns consist of 270 bars and 6 reactions and that the number of joints equals 138; hence, the structure complies with the requirement that twice the number of joints should equal the number of unknowns. Further investigation of the truss both analytically and by means of a model shows that it is stable.

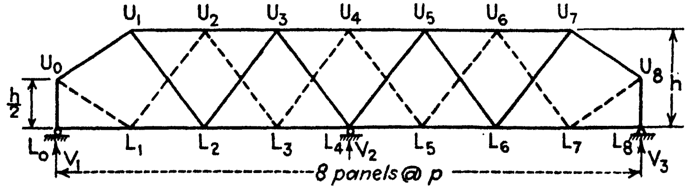

FIG. 23.—Rhomboid truss.

An obvious method of solving the foregoing truss for any loading is to determine the equal horizontal components in bars $L_{11\text{-}12}$ and $L_{12\text{-}13}$ in terms of the vertical reaction at L_0 and from their combined vertical components determine the vertical reaction at L_{12} in terms of the vertical reaction at L_0. With the value at L_{12} thus known, a similar process will give the reactions at

L_{34} and L_{56} also in terms of L_0; hence, there would remain but two unknown vertical reactions which can be solved by the usual methods. Influence lines for the various reactions of this bridge as determined by the preceding method are given in Fig. 22.

The rhomboid truss shown in Fig. 23 is another example of a statically determinate continuous truss provided it extends over two spans only. A symmetrical two-span truss of this type has the peculiar characteristic of having its center reaction due to a single load equal numerically to that load and acting alternately up and down, depending upon the panel point at which it is located. The correctness of this conclusion may be readily seen by dividing the web into two systems as indicated by the light and heavy lines in Fig. 23. Evidently for load W applied at either L_5 or L_7 the vertical component in bar $U_0L_1 = W - \frac{V_3}{2}$, and the horizontal component $= \left(W - \frac{V_3}{2}\right)2\frac{p}{h} = \frac{p}{h}(2W - V_3)$.

Also, by taking moments about U_1, the horizontal component in U_0L_1 is found to equal $V_1(p/h)$

Therefore

$$V_1 = 2W - V_3$$

whence

$$V_1 + V_3 = 2W$$

Applying $\Sigma V = 0$ to entire structure gives

$$V_1 + V_2 + V_3 - W = 0$$

Therefore

$$2W + V_2 - W = 0 \qquad \text{and} \qquad V_2 = -W$$

Also, for a load W applied at L_6 the horizontal component in U_0L_1 equals value previously found with the W term omitted, *viz.*,

$$-V_3\left(\frac{p}{h}\right)$$

But the horizontal component in $U_0L_1 = +\frac{V_1p}{h}$

Therefore

$$V_1 = -V_3$$

whence by applying $\Sigma V = 0$ we obtain $V_2 - W$

Therefore for this case

$$V_2 = W$$

For a symmetrical truss with more panels, the value of the centre reaction due to a single concentrated load also equals the load, its direction depending upon whether it is applied to the system containing the bar U_0L_1 or to the other system.

Application of the foregoing conclusions shows that for a two-span rhomboid truss with 10 equal panels the gross center reaction due to a uniform load over the entire span applied to the bottom chord is upward and equal to the center panel load.

In order that trusses of the type shown in Fig. 19 may be erected by cantilevering without the use of anchorage members, temporary members are needed to prevent the truss from collapsing. For example, in the trusses shown in Fig. 21, in order that any portion of span 15 may be erected by cantilevering out from pier 15, the structure could be made stable during erection by the insertion either of a vertical at panel point 12 or of a chord member between bottom panel points 11 and 13. The additional cost of such members increases the cost of trusses of this type.

It should be noted that if such members are permanent, the trusses would be continuous trusses; moreover, it is quite possible that the size of these members would be determined by erection conditions even if used as permanent bars.

It should also be observed that the only essential structural difference between the truss of the type illustrated by Figs. 19 and 21 and that shown in Fig. 18 consists in the omission of the verticals over the main piers of the truss in Fig. 18. If these verticals should be made light-tension members unable to carry any appreciable stress, the truss of Fig. 18 would be practically identical, so far as the theoretical computations are concerned, with the truss in Fig. 19.

22. Economic Advantages of Statically Indeterminate Continuous Structures.—Bridges containing continuous trusses and girders of the statically indeterminate type are generally found to require less material than equivalent end-supported spans and are often more economical to erect. An exact comparison can be secured only by competitive bids on structures representing the two types designed by the same engineer and under identical specifications. An illustration of the possible difference is shown by two 270-ft. spans of the Lake Champlain Bridge, one end-supported, the other being one span of a two-span continuous bridge the second span of which has a length of 225 ft. For the case of the continuous structure, the total steel in the 270-ft. trusses of the continuous spans including bearings is 563,352 lb., as compared with 656,673 lb. in the end-supported span. No information is available as to the difference in weight between the continuous bridge and two single-span bridges having the same total length as the continuous span.

In the *Engineering News-Record*, Nov. 21, 1935, an article by George W. Lamb, senior designer, Kansas Highway Commission, gives the saving in structural steel in two continuous bridges as compared with end-supported spans as 10 and 19 per cent, respectively. These bridges had the following spans:

Bridge 1: three-span continuous, 128, 160, 128 ft.
Bridge 2: five-span continuous, equal 90-ft. spans.

Problems

1. Compute by the moment-area method the reactions on the beam shown in Fig. 3 if the centre span equals 20 ft., and if I for this span equals $2I$ for the other spans.

2. Compute the reactions for the beam given in Prob. 1 using the analytical method given in Art. 7.

3. Determine graphically by the moment-area method the reactions on a beam with two spans of 10 ft. each loaded with equal concentrated loads P applied at center of each span. The moment of inertia of each span of the beam varies from centre to end as the square of the girder depth, the latter varying as a straight line. The centre depth = 30 in., and end depth = 15 in.

4. Compute by the method given in Art. 7 the reactions and end moments on a single-span beam, fixed at each end, due to a load P at centre of span.

5. Compute by the use of the three-moment equation the reactions and end moments on the beam given in Prob. 4.

6. Using following data, compute ordinates to influence line for reaction at L_{10}, for load at assigned panel points, of the truss shown in Fig. 16. Carry results to three decimal places. For values of a, b, $a^2 - b^2$, and $\frac{a}{a^2 - b^2}$, see Art. 12.

Unit load at panel point	1	2	3	4	5	6	7	8	9
m	−3.87	−7.60	−10.95	−13.44	−14.80	−14.40	−12.36	−9.13	−4.95

Unit load at panel point	11	12	13	14	15	16	17
n	−7.84	−13.27	−17.26	−19.04	−18.48	−17.63	−16.46
p	−3.49	− 6.63	− 9.58	−11.66	−13.54	−15.05	−16.46

Unit load at panel point	18	19	20	21	22	23
n	−15.05	−13.54	−11.66	− 9.58	− 6.63	−3.49
p	−17.63	−18.48	−19.04	−17.26	−13.27	−7.84

7. Compute stresses in assigned bars of trusses shown in Fig. 16 due to differential settlement of 1 in. at either L_0 or L_9 as assigned.

8. Same as Prob. 7, but use truss shown in Fig. 18.

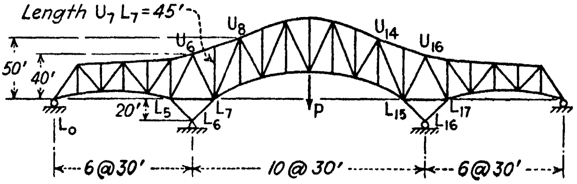

Figure for Prob. 9.

9. *a.* Show that this structure is statically indeterminate to the second degree.

b. Assume bars L_6L_7 and $L_{15}L_{17}$ to be redundant members, and determine magnitude and direction of all reactions due to load shown, assuming stress in each of these bars to be zero.

c. Assume stresses in bar L_6L_7 and $L_{15}L_{17}$ to equal $+X$ and $+Y$, respectively, and compute stress in bar U_6U_8 and $U_{14}U_{16}$ in terms of P, X, and Y.

d. Assume all bars except U_6U_8 and $U_{14}U_{16}$ to be rigid, *i.e.*, to have E = infinity; and determine value and direction of reaction at L_0. Note that bars U_6U_8 and $U_{14}U_{16}$ are identical in length and cross-section area.

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CHAPTER II

ARCHES. GENERAL

23. Classification and Definition.—Structures may be divided into three broad groups in accordance with the direction of the end reactions due to loads acting normally to the line connecting the points of application of these reactions, *viz.*:

A. Structures with parallel reactions.

B Structures with converging reactions.

C. Structures with diverging reactions.

The arch lies in class *B*, the suspension bridge in class *C*, and practically all other structures in class *A*. An arch may therefore be defined as a structure or structural member the end reactions of which converge when the structure is subjected to loads acting normally to the line connecting the points of application of the end reactions. The arch rib itself may be curved or may be composed of straight members. An arched structure such as shown in Fig. 2 (Chap. V) has, as a whole, parallel reactions under normal loads and hence comes under class *A*, but the curved member comes under class *B*.

24. Historical.—The arch is one of the oldest forms of construction, masonry arches having been used for over 5000 years, if we can rely upon the date ascribed to the brick arches recently excavated by archaeologists at Ur near Bagdad as illustrated in Plates V and VI. False arches of corbel construction were frequently used by the ancient Egyptians, but those at Ur are the oldest known examples of true arches. It remained for the Romans, however, to use the arch extensively and many examples of Roman arch construction are in existence, such, for example, as the aqueducts on the outskirts of Rome, the Cloaca Maxima in Rome, and the Pont du Gard in southern France, the last being perhaps the finest example of Roman arch construction in existence.

During the dark ages following the decline of the Roman Empire, the art of construction languished but was revived again following the proclamation of the Council of Arles in 1034 "for the reestablishment of peace and the restoration of the Christian faith." One of the first examples of the revival of arch bridges is shown by the Saint Benezet Bridge across the Rhone at Avignon in southern France. This bridge is often said to have been built by a religious order entitled the Brothers of the Bridge, but careful research in recent years has shown no real evidence of a unified religious order devoted to bridge construction, although it is probable that the Saint Benezet Bridge and the Pont Saint Esprit, also crossing the Rhone about 25 miles north of Avignon, and London Bridge, which was built at about the same period, were all constructed under the direction of fervent Christian monks and laymen, who were inspired by their faith to aid pilgrims and merchants to cross in safety dangerous and otherwise nearly impassable rivers. The tradition of the Order of the Bridge Builders is perhaps due in some measure to the fact that Pontifex Maximus, one of the titles of the pope, may be translated as chief bridgemaker. This title, however, was that given to the head of one of the great priestly Roman orders, thus antedating the order of the so-called pontist brothers; and one authority, at least, considers that the word *pons* originally meant "path" rather than "bridge," especially the path between man and the gods, and that the Pontifex Maximus kept this path open.

The earlier arches were all of masonry, the stone-arch age extending to the nineteenth century. During the last century the development of iron and steel making and of reinforced concrete has made it possible to extend the scope of arch construction from the 300-ft. maximum spans of stone arches to the 1550-ft. spans of steel arches such as cross the Kill van Kull in New York City and the harbour at Sydney, New South Wales.

As usual, the art of construction preceded the theory, and it was not until the beginning of the present century, 44 centuries after the construction of the first true arch known, that the elastic theory of arch construction was developed and extensively applied. It is this theory that is presented in this book.

25. Types of Arches.—Up to a comparatively recent period the arch was always constructed as a statically undetermined structure, similar to that shown in Fig. 1, which represents the conventional masonry arch with reactions fixed in neither direction, magnitude, nor point of application, the arch being in consequence

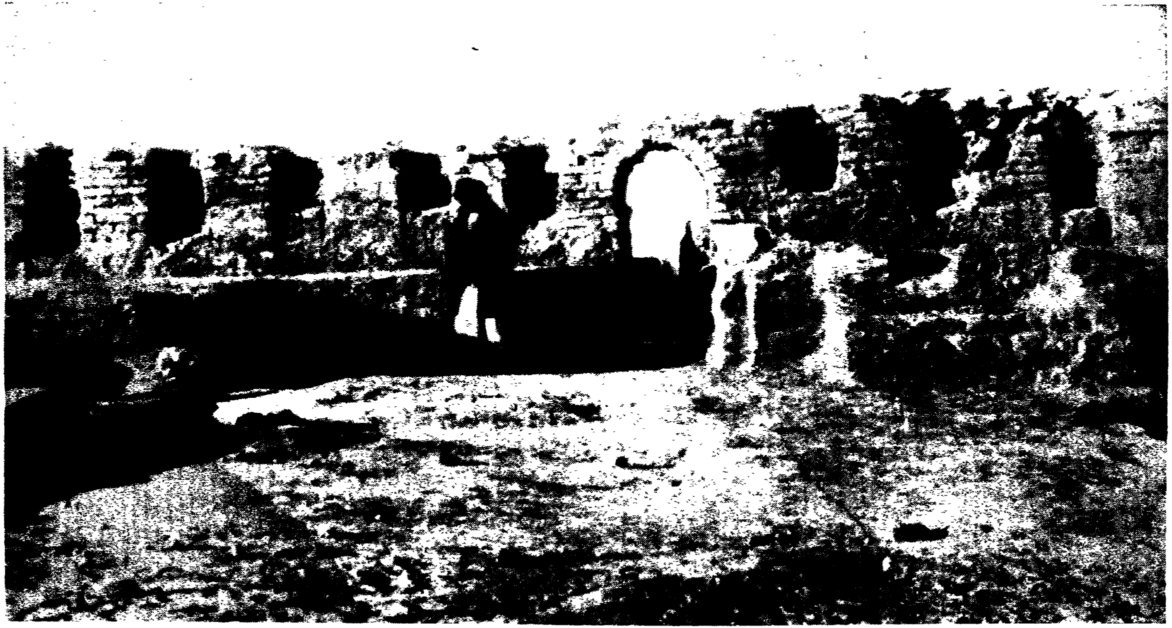

Plate V.—Ancient arch. Temple of the moon god. Ur, near Bagdad. Constructed about 1500 B.C. (*Courtesy of the University Museum, Philadelphia.*)

PLATE VI.—Ancient Sumerian arches. Ur, near Bagdad. Constructed about 3500 B.C. View showing arched door of grave and arched roof of the chamber. (*Courtesy of the University Museum, Philadelphia.*)

statically undetermined in a threefold degree, having six unknowns.

With the application of iron and steel to bridge construction came a recognition of the advantage of statical determination, and metal arches began to be constructed in which some of the unknowns were eliminated by the insertion of hinges. Such arches are shown in Figs. 2 and 3. If in the arch shown in Fig. 3 a hinge be inserted at the centre similar to that of the arch shown in Fig. 2, the arch becomes a three-hinged arch and is statically determined.

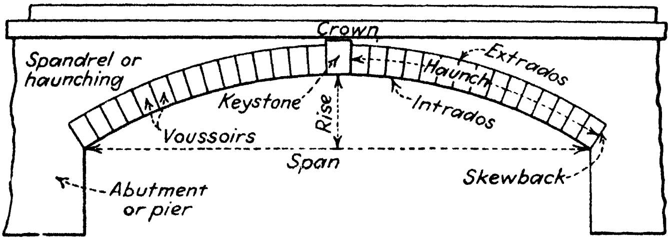

FIG. 1.—Masonry arch.

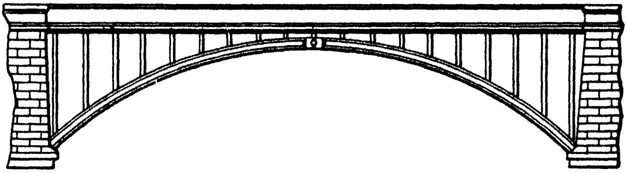

FIG. 2.—Metal arch with one hinge.

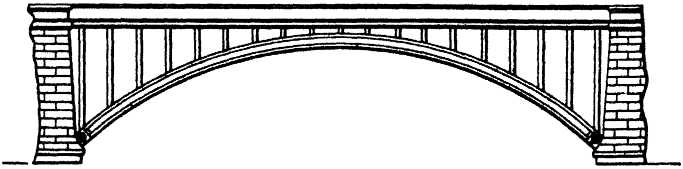

FIG. 3.—Two-hinged metal arch.

The ribs of metal arches may be formed of I beams, of plates and angles as in plate girders, of cast-iron or cast-steel segments bolted together, or of riveted trusses.

In recent years a considerable number of three-hinged masonry arches have been constructed, and following the application of plain and reinforced concrete to arch design, the custom of applying the loads to the arch rib at fixed points by transverse walls has also been adopted in many long-span bridges, thus doing away with some of the uncertainty that formerly occurred in such cases and securing many of the advantages of the metal arch. Figure 4 illustrates such an arch. It should be said, however, that the common type of masonry arch now used is that without hinges, shown by Fig. 1, with the voussoirs replaced by a continuous rib of reinforced concrete.

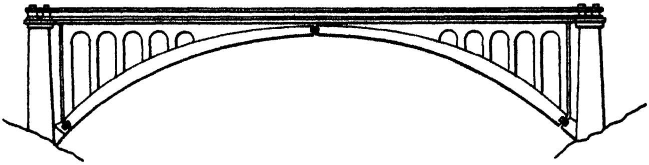

FIG. 4.—Three-hinged masonry arch.

An important type of metal arch—the spandrel-braced arch—is shown by Fig. 5. This structure is in reality a combination of a truss and an arch rib. If the arch rib of the three-hinged, spandrel-braced arch be constructed to a parabolic curve, the diagonals and top chord will not be in action under a full uniform load, the arch rib in that case acting like the arches previously described, the floor loads being applied through the vertical posts.

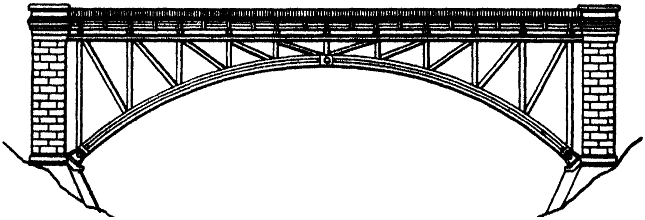

FIG. 5.—Three-hinged spandrel-braced metal arch.

Like other arches, the spandrel-braced type is frequently constructed as a two-hinged arch.

Another type of arch structure in occasional use is that of an arched rib connected by tie rods as illustrated by Fig. 2 (Chap. V).

26. Determination of Span and Rise.—The span and rise of any given arch are generally determined by the conditions of the problem. If the arch is one of a series, the relative length and rise of each span as compared with those of the other spans are particularly important from the standpoint of appearance and

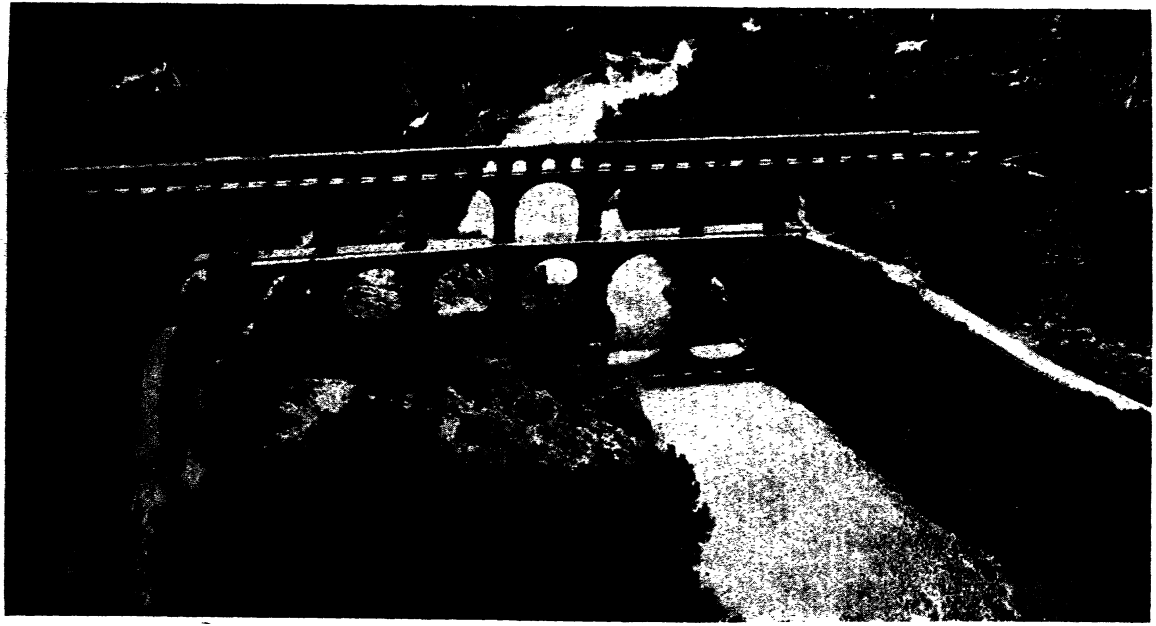

PLATE VII.—Pont du Gard near Avignon, France. Roman arch constructed about 20 B.C.

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PLATE VIII.—Pont St. Benezet at Avignon, France. Constructed about 1177–1185.

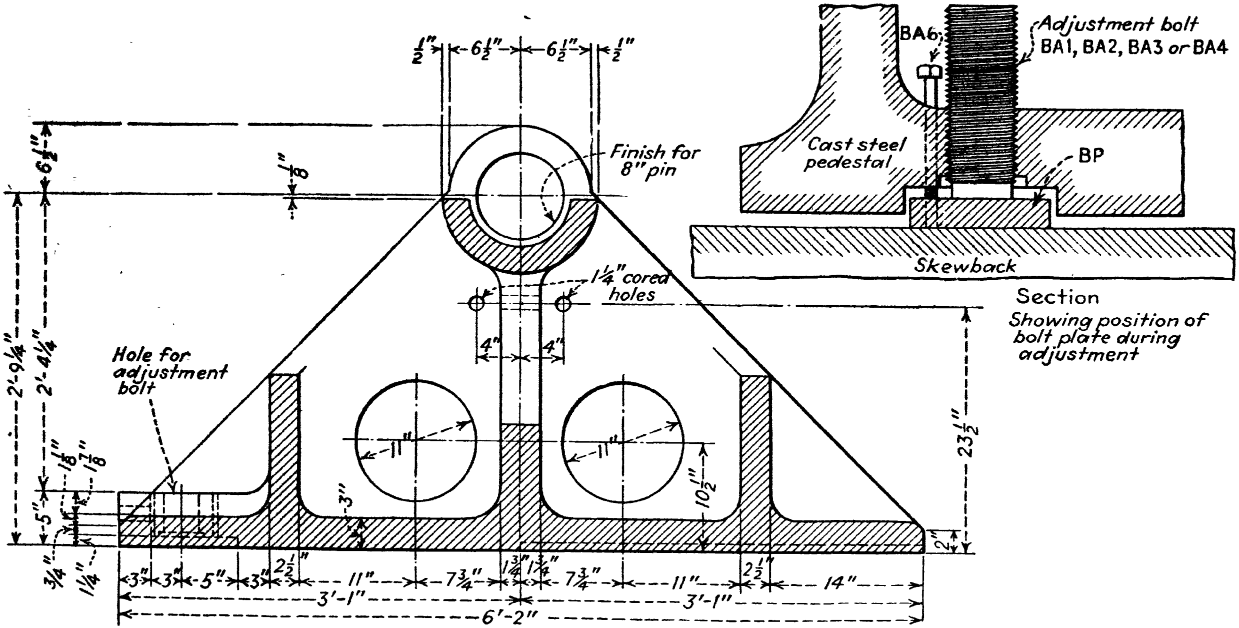

FIG. 6.—Section of shoe casting for hinge used on Hampden County Memorial Bridge, Springfield, Mass.

must be given careful consideration. If the arch is over a stream, adequate waterway and headroom must be provided. An arch span of the ordinary type should not be used over streets unless considerable headroom and rise of arch can be obtained; otherwise, the arch gives an appearance of clumsiness and oppression rather than of grace and freedom. For such crossings with low headroom, a girder span or a structure of the so-called rigid-frame type is more satisfactory.

27. Hinged-arch Ribs. General.—The three-hinged arch is statically determined; methods of determining the reactions are discussed in Chap. X, 3d ed., "The Theory of Structures," Spofford, and will not be given here.

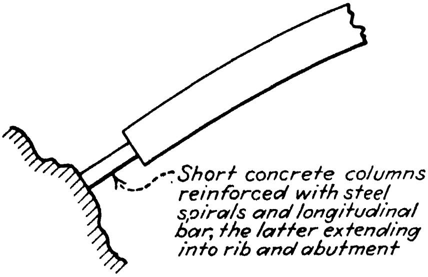
Short concrete columns reinforced with steel spirals and longitudinal bar; the latter extending into rib and abutment

FIG. 7.

The two-hinged arch is statically undetermined to the first degree, since there are two unknown reaction components at each hinge, or four in all, *i.e.*, one more than the number of statical equations available for solution.

The spandrel-braced, two-hinged trussed arch can be readily computed by the method of least work, using the equations given in "Theory of Structures," Spofford, and the methods applied in detail to continuous trusses as illustrated in Chap. I of this book.

The two-hinged ribbed arch is considered at length in this book.

Hinges may be used for either masonry or steel ribs, although they are not so common for masonry as for steel structures. The hinges may be constructed in a manner similar to the construction of the end supports of ordinary bridge trusses, *i.e.*, by the use of a bridge pin bearing upon either structural- or cast-steel pedestal, as illustrated by Fig. 6, which represents one of the end hinges of the heavy reinforced-concrete arch ribs used in

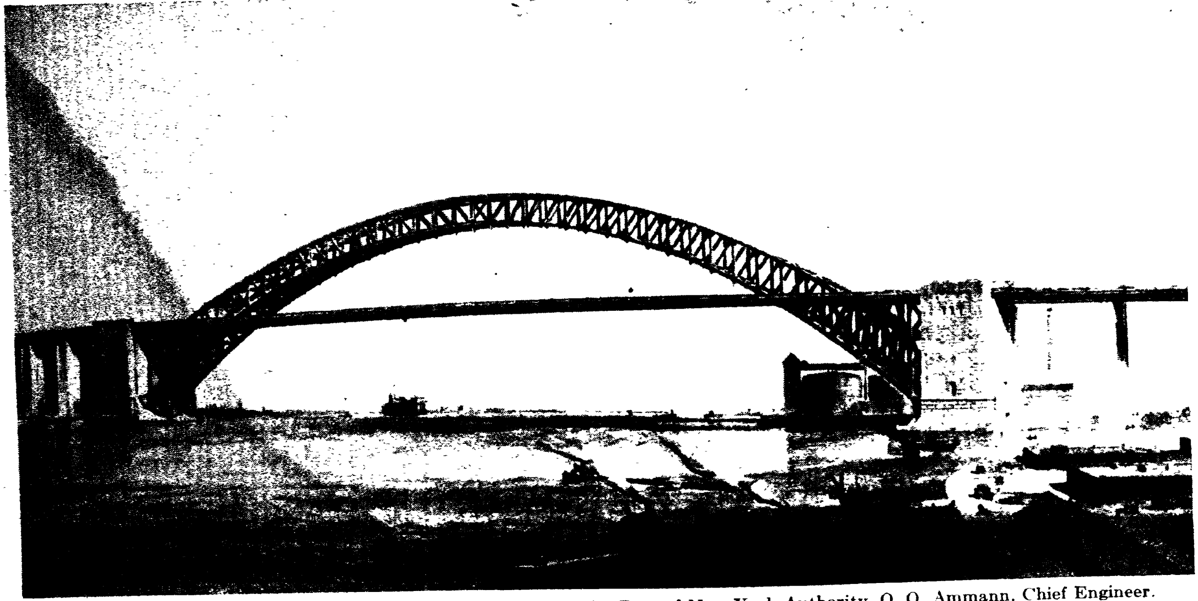

Plate IX.—Bayonne Bridge, across the Kill van Kull, built by Port of New York Authority, O. O. Ammann, Chief Engineer.

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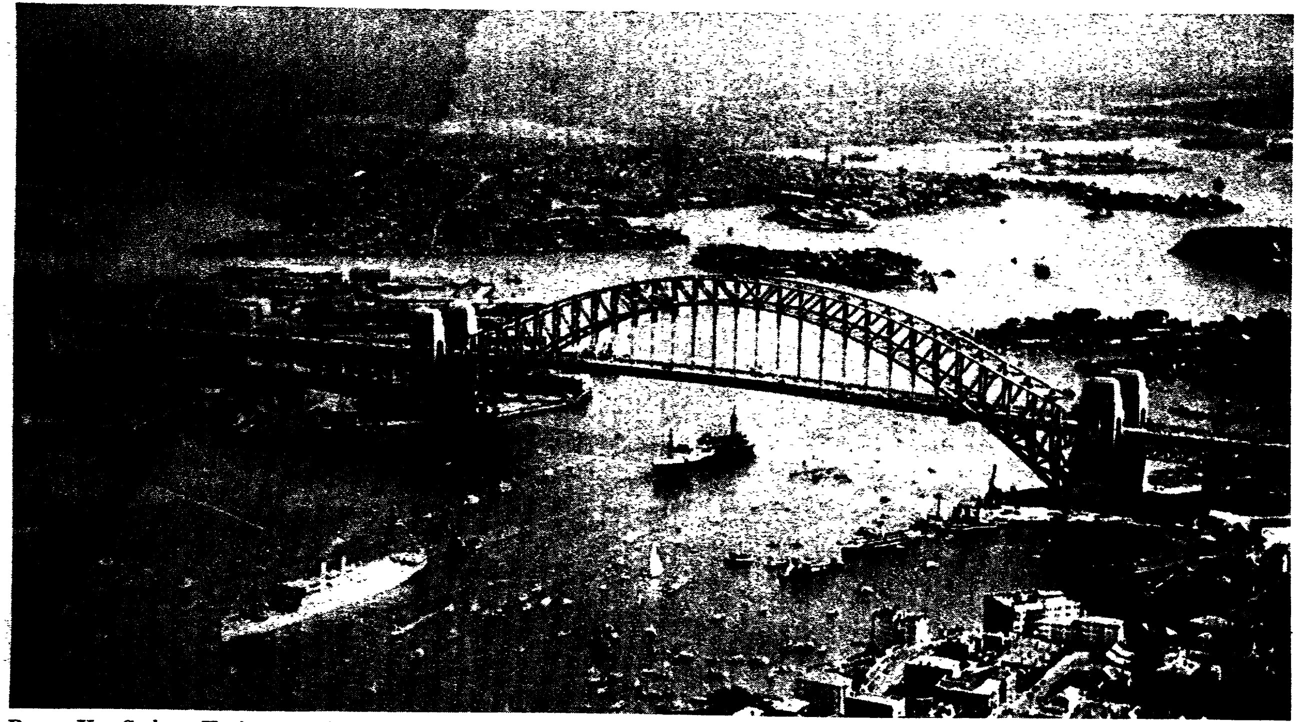
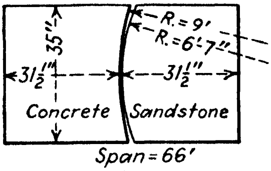
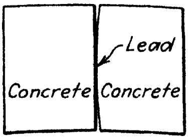


Plate X.—Sydney Harbour Bridge, New South Wales. Dr. John Job Crew Bradfield, Principal Designing Engineer, Public Works Department of New South Wales. (*Courtesy of the New South Wales Government.*)

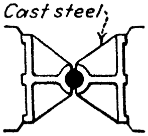
the Hampden County Memorial Bridge across the Connecticut River at Springfield, Mass. In concrete arches, hinges such as shown in Fig. 7 have been used during erection and later closed in to make fixed ends. Other types of hinges that have been used for reinforced-concrete arches are illustrated by Figs. 8 and 9.[1]



Hinge at Abutment



Hinge at Crown



Crown

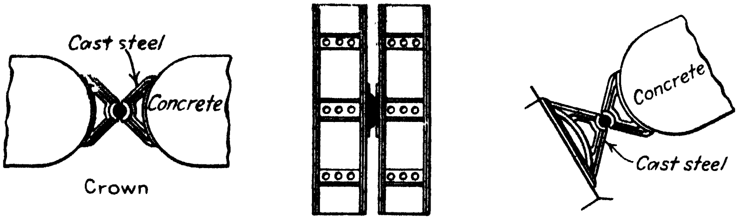


FIG. 8.

While three-hinged arches have the advantage from the standpoint of statical determination, they are less rigid than two-hinged arches and require an expansion joint at the centre; they are, in consequence, seldom used for the completed structure in the case of bridges carrying heavy live loads. The use of an intermediate hinge during erection to relieve the rib from stresses caused by settlement as a result of poor foundations is, however, a common practice, the hinge being eliminated after a considerable portion, or the whole, of the dead load is in place—in concrete arches by the filling of the joint with concrete, and in steel arches by riveting the halves together.

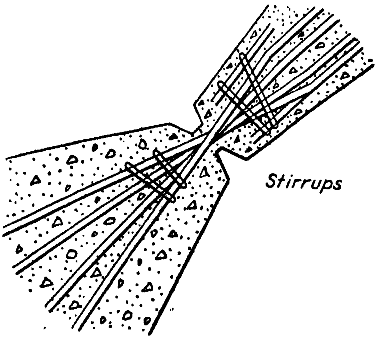


FIG. 9.

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[1] See also references at end of chapter.

Hinged-arch ribs as used in bridges are generally of parabolic outline in order that the line of resistance for the dead load, which is approximately uniformly distributed, may coincide with the centre axis of the arch, thus minimizing the moment due to dead load. If the arch is comparatively flat, the arch rib is generally made of constant cross section.

Segmental arches are seldom used for bridges, but inasmuch as they are susceptible, if of uniform cross section, to precise analysis, they are treated fully in Chap. VI.

The reader should note before considering the analytical methods that follow that the vertical reactions due to a vertical load for a symmetrical hinged arch with supports at same level are identical with the corresponding reactions for a horizontal beam under the same load; hence, the computations for such a structure may, if desired, be divided into two cases as follows:

1. End-supported beam under vertical loads.
2. Arch rib subjected to horizontal forces only.

28. Preliminary.—In order to make a thorough study of arch action it is necessary to consider first the distribution of normal stress over the cross section of any bar, straight or curved, and the following chapter is therefore devoted to such consideration.

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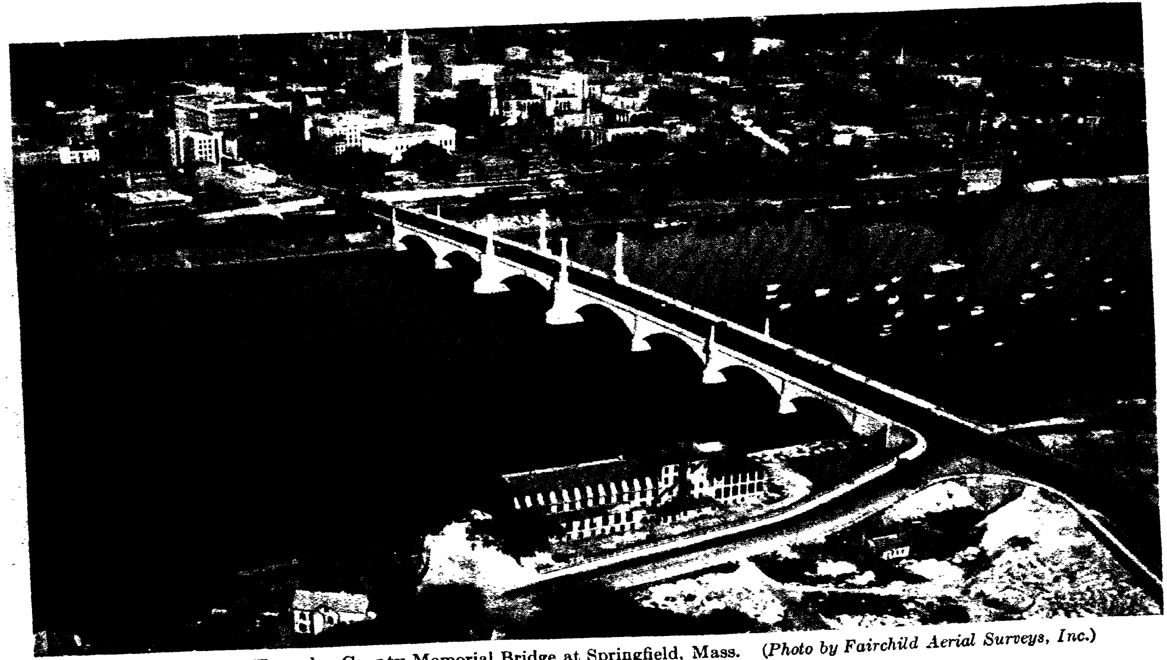

Plate XI.—The Hampden County Memorial Bridge at Springfield, Mass. (*Photo by Fairchild Aerial Surveys, Inc.*)

(*Facing page* 68)

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CHAPTER III

DISTRIBUTION OF NORMAL STRESSES ON CROSS SECTIONS OF STRAIGHT AND CURVED BARS

29. General.—In the treatment of straight bars the discussion will be limited to the case of a straight bar of uniform cross section subjected to loads acting in a plane containing the longitudinal axis of the bar, *i.e.*, the axis passing through the centroids of all cross sections. If, when the loads produce shear on the cross sections of the bar, the plane of loading does not also contain the longitudinal axis through the centres of twist of the cross sections,[1] the bar will be subjected to torsion—a state that would not affect the normal stresses, should it exist. This chapter does not include any discussion or applications involving torsion.

In the case of both straight and curved bars the fundamental assumptions of the ordinary beam formula will be made, *viz.:*

1. Transverse sections which are plane before bending remain plane after bending.
2. Material is homogeneous, obeys Hooke's law, and stress does not exceed the elastic limit.
3. Every layer of material is free to expand or contract longitudinally and laterally under stress as if entirely separate from other layers.
4. Modulus of elasticity is the same for tension and compression.
5. Fibres that are parallel to the central axis before the bar is bent remain so after bending; *i.e.*, a section at right angles to the central axis before bending remains at right angles after bending.

30. Neutral Axis. Definition.—The neutral axis of a section of a stressed bar is the locus of those points in the plane of the section at which the normal fibre stresses are zero. In the case of a straight beam subjected only to a bending moment applied in the plane of one of its principal axes, the neutral axis of any section passes through its centroid and coincides with the principal axis perpendicular to the plane of loading. That the neutral axis is a straight line is evident, provided Hooke's law holds.

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[1] Strength of Materials, Part 1, p. 191, Art. 41, Timoshenko. D. Van. Nostrand Company, Inc., New York. 1930.

31. Position and Slope of Neutral Axis of a Straight Bar.—The position and slope of the neutral axis of the cross section of a straight bar loaded with a force parallel to the axis of the bar depends upon the point of application of the load. If the force be applied at any point between the centroid and the edge of the cross section, the neutral axis will not pass through the centroid and may or may not pass through the cross section. If the force be applied at the centroid of the section, the neutral axis will lie at infinity. The slope of the neutral axis depends upon the position of the point of application of the load, and it may or may not coincide or be parallel with one of the principal axes of the cross section. The foregoing conclusions are applicable also to a member subjected to a bending moment and an axial force combined, since these may be replaced by an axial force acting at some point other than the centroid of the cross section.

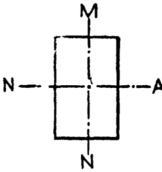


FIG. 1.

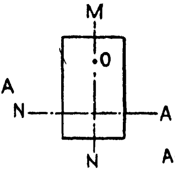


FIG. 2.

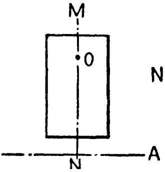


FIG. 3.

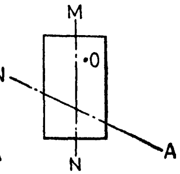


FIG. 4.

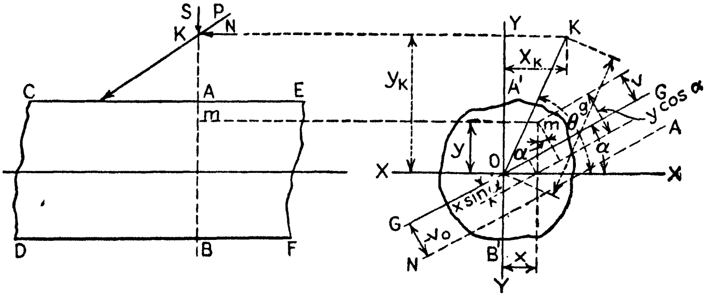


FIG. 5.

The foregoing conditions are illustrated for simple cases by Figs. 1 to 4. In all the figures *MN* is a principal axis, and *O* is the point of application of the load. Figure 1 is the common case of a straight beam subjected to pure flexure only. Figure 2 is a column with a load not at the centroid but applied in one of the principal axes. Figure 3 is a beam subjected both to flexure and to a direct force applied in one of the principal axes. Figure 4 is a column with the load applied in neither of the principal axes of the cross section. For this case the neutral axis is not parallel to either principal axis.

In Fig. 5 let *CEDF* be a straight bar of homogeneous material subjected to a force P lying in a plane containing the longitudinal axis of the bar, and let the problem be to determine the stress at any point m of the cross section AB.

Let the force P be resolved at the point K, where the force cuts the plane of the cross section, into two components S and N tangential and normal, respectively, to the given section. The tangential component produces shear only and will be neglected in this discussion. The normal component causes direct stress N and bending moment Ng on the cross section.

Let XX and YY be *any* pair of rectangular axes, passing through the centroid O of the figure.

A = area of cross section in square inches.

I_x = moment of inertia about axis XX in in.^{4}

I_y = moment of inertia about axis YY in in.^{4}

J = the product of inertia $\int\int xy dx dy$ in in.^{4}

g = distance from K to O.

f = intensity of stress at point m (+ when tension).

f_o = intensity of stress at any point on line GG.

Let x_k and y_k be the coordinates of the point K, and x and y the coordinates of the point m, and let NA represent the neutral axis which makes an angle α with XX. Let y and y_k be positive when above XX and x and x_k when to right of YY.

Let GG be drawn through the centroid O parallel to NA, and let the distance between these two axes be v_o.

By making the usual assumptions that cross sections remain plane after bending and that stress varies as strain, and by considering that v_o is negative for the assumed position of NA, the following equation may be written for the value of f at point m:

$$f = f_o - \frac{f_o v}{v_o} \tag{1}$$

The value of v referred to the axes XX and YY is given by the following equation:

$$v = y \cos \alpha - x \sin \alpha$$

Substituting this value in Eq. (1) gives the following equation:

$$f = f_o - \frac{f_o}{v_o}(y \cos \alpha - x \sin \alpha) \tag{2}$$

Substituting this value of f in the familiar equations of mechanics, noting that $\int\int xdxdy$ and $\int\int ydxdy$ each $= 0$, gives

$$N = \int\int fdxdy = f_oA \tag{3}$$

$$\begin{aligned} Nx_k &= \int\int fxdxdy = \int\int\left[f_o - \frac{f_o}{v_o}(y \cos \alpha - x \sin \alpha)\right]xdxdy \\ &= -\frac{f_o}{v_o}(J \cos \alpha - I_y \sin \alpha) \end{aligned} \tag{4}$$

$$\begin{aligned} Ny_k &= \int\int fydxdy = \int\int\left[f_o - \frac{f_o}{v_o}(y \cos \alpha - x \sin \alpha)\right]ydydx \\ &= -\frac{f_o}{v_o}(I_x \cos \alpha - J \sin \alpha) \end{aligned} \tag{5}$$

Whence

$$\frac{Nx_k}{J \cos \alpha - I_y \sin \alpha} = \frac{Ny_k}{I_x \cos \alpha - J \sin \alpha}$$

But

$$\frac{y_k}{x_k} = \tan \theta$$

Therefore

$$\tan \theta = \frac{I_x \cos \alpha - J \sin \alpha}{J \cos \alpha - I_y \sin \alpha}$$

from which we may derive each of the following equations for α:

$$\tan \alpha = \frac{I_x - J \tan \theta}{J - I_y \tan \theta} \tag{6a}$$

$$= \frac{I_x \cot \theta - J}{J \cot \theta - Iy} \tag{6b}$$

$$= \frac{I_x \cos \theta - J \sin \theta}{J \cos \theta - I_y \sin \theta} \tag{6c}$$

J may be computed for any pair of rectangular axes by use of the following equation:

$$J = J_0 + Akh$$

In this equation J_0 = product of inertia for any given pair of rectangular axes passing through centroid.

k and h = the coordinates of the origin of the new pair of axes referred to the pair for which J_0 is computed.

For a pair of axes one of which is an axis of symmetry, $J = 0$.

It may also be noticed that the angle ϕ between any horizontal axis and the axis about which I is either a maximum or a minimum is given by the equation

$$\tan 2\phi = 2\int\int \frac{xydxdy}{I_y - I_x} = \frac{2J}{I_y - I_x} \tag{7}$$

If $J = 0$ as for the case when one of the axes is an axis of symmetry, ϕ has its minimum value; hence either the axis XX or YY is an axis of symmetry.

32. Intensity of Stress at Any Point of the Cross Section of a Straight Bar.—Substituting in Eq. (2) the value of f_o and f_o/v_o from Eqs. (3) and (4),

$$f_o = \frac{N}{A} \qquad \text{and} \qquad \frac{f_o}{v_o} = -\frac{Nx_k}{J \cos \alpha - I_y \sin \alpha}$$

gives the following equation for f:

$$f = \frac{N}{A} + \frac{Nx_k(y - x \tan \alpha)}{J - I_y \tan \alpha} \tag{8}$$

Substituting in Eq. (6a) the value of $\tan \theta$, *viz.*, y_k/x_k, gives the following expression:

$$\tan \alpha = \frac{I_x - J\dfrac{y_k}{x_k}}{J - I_y\dfrac{y_k}{x_k}}$$

which, when substituted in Eq. (8), gives the following:

$$f = \frac{N}{A} + \frac{N[(y_kI_y - x_kJ)y + (x_kI_x - y_kJ)x]}{I_xI_y - J^2} \tag{9}$$

Equation (9) may be written thus:

$$f = \frac{N}{A} + Nx_k\frac{Jy - I_x x}{J^2 - I_x I_y} + Ny_k\frac{Jx - I_y y}{J^2 - I_x I_y} \tag{10}$$

If one of the axes is an axis of symmetry, $J = 0$; hence for this case

$$f = \frac{N}{A} + Nx_k\frac{x}{I_y} + Ny_k\frac{y}{I_x} \tag{11}$$

If M_y and M_x be substituted for Nx_k and Ny_k, respectively,

$$f = \frac{N}{A} + \frac{M_y x}{I_y} + \frac{M_x y}{I_x} \tag{12}$$

The signs of the various quantities in Eqs. (8) to (12), inclusive, should be carefully observed in applying these equations. If N is positive when tension, x positive when to the right of axis YY, and y positive when above axis XX, then f will be tension if the foregoing equation gives a positive value. M_x and M_y are positive when they exert the same effect as a positive value of the product Nx_k or Ny_k, respectively.

Equation (12) is a general expression for the normal stress at any point m in the cross section of a symmetrical member in which the coordinate axes XX and YY are principal axes of the cross section due to a load applied at any point k. If the load be applied in either of the principal axes, one of the following familiar formulas is obtained:

$$f = \frac{N}{A} + \frac{M_x y}{I_x} \qquad \text{or} \qquad f = \frac{N}{A} + \frac{M_y x}{I_y}$$

If in Eq. (9) we substitute for y_k and x_k their respective numerical values $g \sin\theta$ and $g \cos\theta$, the following equation results:

$$f = \frac{N}{A} + Ng\frac{(I_y \sin\theta - J\cos\theta)y + (I_x \cos\theta - J\sin\theta)x}{I_x I_y - J^2} \tag{13}$$

which may be written

$$f = \frac{N}{A} + \frac{Ng}{S} \tag{14}$$

or

$$f = \frac{N}{A} + \frac{M}{S} \tag{15}$$

in which S is the *section modulus* in its most general form, *i.e.*, for any point m for which the stress due to a force applied at any point k is to be determined.

In determining the value of S, careful attention must be given to the signs not only of x and y but also of $\cos\theta$ and $\sin\theta$. This is illustrated by the following case:

Assume a symmetrical section in which case $J = 0$; then

$$S = \frac{I_xI_y}{(I_y \sin\theta)y + (I_x \cos\theta)x}$$

If

$$\theta = 0°, \qquad S = \frac{I_xI_y}{I_xx} = \frac{I_y}{x}$$

If

$$\theta = 180°, \qquad S = \frac{I_xI_y}{I_x(-1)x} = -\frac{I_y}{x}$$

For the case considered, Ng is the bending moment due to the eccentricity of the load. It equals the moment of a couple composed of two forces N, one applied at the point K, and the other applied at the centroid of the cross section. It is evidently immaterial whether the actual moment on the cross section is caused by this couple or by any other set of forces acting in the given plane KO. In other words, Eqs. (14) and (15) may be used to find the fibre stress at any point m of a cross section subjected to a normal force N and a bending moment M, it being immaterial whether this moment is due to an eccentric load, to a moment caused by forces acting at right angles to the axis of a bar as in the case of a beam, or to both together.

33. The S Polygon for a Straight Bar.—From Eq. (13) the value of S is found to be

$$S = \frac{I_xI_y - J^2}{(I_y \sin\theta - J\cos\theta)y + (I_x\cos\theta - J\sin\theta)x} \qquad (16)$$

This value may be expressed as a linear equation by substituting for $S \sin\theta$ and $S\cos\theta$ the terms z and q, respectively. This is equivalent to plotting S as a lever arm making an angle θ with the horizontal and using z and q for its coordinates as shown in Fig. 6.

If this is done, and if the terms x and y now in the equation are called x_m and y_m, the coordinates of a definite point m in the cross section for which the stress is to be determined, the following equation results for the relation between z and q:

$$I_xI_y - J^2 = y_m(zI_y - Jq) + x_m(I_xq - Jz)$$

from which may be derived the linear equation

$$z = q\frac{Jy_m - I_xx_m}{I_yy_m - Jx_m} + \frac{I_xI_y - J^2}{I_yy_m - Jx_m} \tag{17}$$

The fact that this is a linear equation for any given point shows that the locus of the ends of S when plotted as lever arms with O as a center is a straight line of indefinite length for any given point in the cross section. This locus may be called the S line for the given point. If this line is plotted, the value of S, the section modulus for the given point, may be obtained for a force acting at any point K in the cross section, by measuring the intercept between O and the S line for the given point on a line drawn through K and O or for a bending moment equivalent to that produced by a couple acting in a plane through O by measuring the intercept between O and the S line on a line lying in the plane of action of the couple. This is illustrated in Fig. 7 where the S line is plotted for the corner b of the rectangle $abcd$. For this case the ordinate OS equals the section modulus S, for a force acting at K or a couple acting in the plane OK; hence, the stress at $b = \dfrac{N}{A} + \dfrac{M}{OS}$.

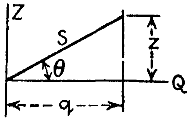


FIG. 6.

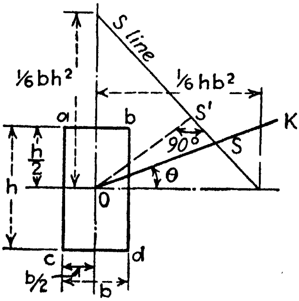


FIG. 7.

FIGS. 6–7.

For this section, since axes are principal axes,

$$y_m = \frac{h}{2} \qquad x_m = \frac{b}{2} \qquad I_x = \frac{1}{12}bh^3 \qquad \text{and} \qquad I_y = \frac{1}{12}hb^3$$

Therefore since $J = 0$

$$z = q\frac{-\frac{1}{12}bh^3\frac{b}{2}}{\frac{1}{12}hb^3\frac{h}{2}} + \frac{\frac{bh^3}{12}}{\frac{h}{2}} = -\frac{h}{b}q + \frac{1}{6}bh^2$$

To locate the S line, determine z where $q = 0$, and q where $z = 0$. For the first case, $z = \frac{1}{6}bh^2$; and for the second case, $q = hb^2/6$.

It is obvious from the figure that for this case the minimum value of S equals the perpendicular distance from O to the S line; hence the maximum possible stress at b would be produced by a load or a moment acting in the plane OS'. It is also obvious that if the plane of action of the forces should be parallel to the S line, the section modulus would be infinite, in which case the stress at point b due to flexure only would equal zero; *i.e.*, point b for this case is a point on the neutral axis *provided* the section is exposed to pure bending.

When the plane of action of the force in the previous case is in either principal axis, we get the familiar values for the section modulus of $\frac{1}{6}bh^2$, or $\frac{1}{6}hb^2$.

The cross section of any ordinary structural member can be inscribed in a polygon with apices coinciding with points on the cross section where maximum fibre stresses may occur.

By inserting in Eq. (17) values of x_m and y_m for the coordinates of the various apices of the circumscribing polygon, a series of lines can be found forming the S polygon for the section. This will be a closed figure with a side corresponding to each apex of the circumscribing polygon. These sides will be the S lines, intersecting and forming apices of the S polygon whenever θ has a value that will cause two adjacent apices of the circumscribing polygon to be extreme fibres at the same time. This will happen whenever a side of the circumscribing polygon is parallel to an axis of reference. If all dimensions are in inches, S will be in inches^3.

The coordinates of an apex of the S polygon can be determined by substituting successively in Eq. (17) for the distances q and z the coordinates x_{ab} and y_{ab} of the apex of the S polygon corresponding to the side ab, and for x_m and y_m the coordinates x_a, y_a and x_b, y_b of two adjoining apices of the circumscribing polygon, and solving for the intersections as follows. From Eq. (17)

$$q = z\frac{I_y y_m - J x_m}{J y_m - I_x x_m} - \frac{I_x I_y - J^2}{J y_m - I_x x_m}$$

whence

$$y_{ab}\frac{I_y y_a - Jx_a}{Jy_a - I_x x_a} - \frac{I_x I_y - J^2}{Jy_a - I_x x_a} = y_{ab}\frac{I_y y_b - Jx_b}{Jy_b - I_x x_b} - \frac{I_x I_y - J^2}{Jy_b - I_x x_b}$$

Therefore

$$y_{ab}\left(\frac{I_y y_b - Jx_b}{Jy_b - I_x x_b} - \frac{I_y y_a - Jx_a}{Jy_a - I_x x_a}\right) = (I_x I_y - J^2) \left(\frac{1}{Jy_b - I_x x_b} - \frac{1}{Jy_a - I_x x_a}\right)$$

and

$$y_{ab} = \frac{(I_x I_y - J^2)[J(y_a - y_b) - I_x(x_a - x_b)]}{J^2(x_a y_b - x_b y_a) + I_y I_x(y_a x_b - y_b x_a)}$$
$$= \frac{J(y_a - y_b) - I_x(x_a - x_b)}{y_a x_b - y_b x_a} \qquad (18)$$

Also from Eq. (17)

$$x_{ab}\frac{Jy_a - I_x x_a}{I_y y_a - Jx_a} + \frac{I_x I_y - J^2}{I_y y_a - Jx_a} = x_{ab}\frac{Jy_b - I_x x_b}{I_y y_b - Jx_b} + \frac{I_x I_y - J^2}{I_y y_b - Jx_b}$$

whence,

$$x_{ab} = \frac{J(x_a - x_b) - I_y(y_a - y_b)}{x_a y_b - x_b y_a} \qquad (19)$$

If side *bd* is substituted for side *ab*, the foregoing equations are applicable to the determination of the apex *bd* of the *S* polygon by interchanging axes *XX* and *YY* and making the following changes in the coordinate distances:

y_b for x_a
y_d for x_b
x_b for y_a
x_d for y_b
x_{bd} for y_{ab}
y_{bd} for x_{ab}
I_y for I_x
I_x for I_y

Substitution of the foregoing values in Eqs. (18) and (19), respectively, gives the following equations.

From Eq. (18)

$$x_{bd} = \frac{J(x_b - x_d) - I_y(y_b - y_d)}{x_b y_d - x_d y_b} \qquad (18a)$$

From Eq. (19)

$$y_{bd} = \frac{J(y_b - y_d) - I_x(x_b - x_d)}{y_b x_d - y_d x_b} \qquad (19a)$$

If the side is parallel to an axis of reference Eqs. (18), (19), (18*a*), and (19*a*) become much simplified.

If the side *ab* is parallel to XX, $y_a = y_b = y$
Therefore Eqs. (18) and (19) become

$$y_{ab} = \frac{I_x}{y} \tag{20}$$

and

$$x_{ab} = \frac{J}{y} \tag{21}$$

If the side *bd* is parallel to XX in its rotated position,

$$y_b = y_d = y$$

Therefore Eqs. (18*a*) and (19*a*) become

$$x_{bd} = \frac{J}{y} \tag{22}$$

$$y_{bd} = \frac{I_x}{y} \tag{23}$$

Equations (22) and (23) may now be referred to the original axes by substituting y_{bd} for x_{bd}, x_b for y and I_y for I_x, giving

$$x_{bd} = \frac{I_y}{x_b} \tag{24}$$

$$y_{bd} = \frac{J}{x_b} \tag{25}$$

The application of the preceding equations is illustrated by the following simple example. Determine coordinates of apices of the S polygon for the symmetrical rectangle *abcd* shown in Fig. 8.

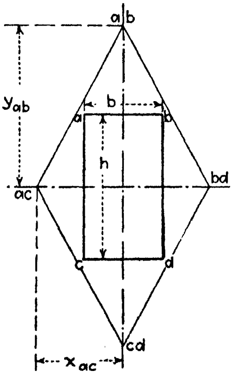


Fig. 8.

For x_{ab} and y_{ab}

$$J = 0 \qquad y_a = y_b = \frac{h}{2} \qquad x_a = -x_b = \frac{b}{2}$$

Therefore from Eqs. (20) and (21)

$$x_{ab} = 0 \qquad y_{ab} = \frac{I_x}{y} = \frac{bh^2}{6}$$

For x_{bd} and y_{bd} either Eqs. (22) and (23) which refer to the rotated axes or (24) and (25) which refer to the axes shown in Fig. 8 may be used. Adopting the latter and substituting $J = 0$, $x = \frac{b}{2}$, $y = \frac{h}{2}$, give the following results:

$$x_{bd} = \frac{I_y}{x_b} = \frac{1}{12}\frac{hb^3}{b/2} = \frac{1}{6}hb^2$$
$$y_{bd} = 0$$

Since the structure shown in Fig. 8 is symmetrical the coordinates of the other apices of the S polygon are evidently located as shown in the figure.

34. S Polygons for Typical Sections.—Coordinates of the S polygons for various structural steel standard sections are as follows:

Case A. Circumscribing polygon of cross section is a rhombus with axes corresponding to axes of symmetry of a symmetrical surface. This condition applies to the extreme corners of rectangular beams, I beams, channel columns with flanges turned in either direction, and other symmetrical figures.

Since $J = 0$ for a symmetrical figure,

$$x_{ab} = 0 \qquad \text{and} \qquad y_{ab} = \frac{I_x}{y_a} = -y_{cd}$$

also

$$y_{ac} = 0, \qquad x_{bd} = \frac{I_y}{x_b} \qquad \text{and} \qquad x_{ac} = \frac{I_y}{x_c}$$

Therefore the S polygon will be as shown in Fig. 8. For a rectangular beam the ordinates of the apices may be written

$$y_{ab} = y_{cd} = \tfrac{1}{6}bh^2$$
$$x_{ac} = x_{bd} = \tfrac{1}{6}hb^2$$

Case B. Circumscribing polygon is a polygon with axes corresponding to principal axes of the surface under consideration, the latter being assumed as symmetrical about one axis only. This case applies to the extreme corners of a channel section as shown in Fig. 9 and to similar unsymmetrical sections.

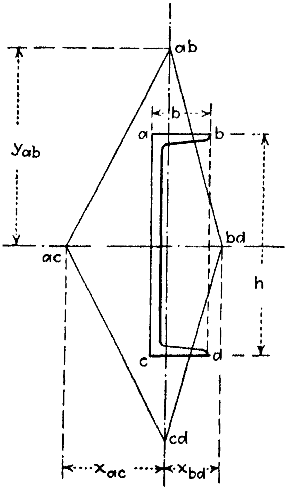


FIG. 9.

For this case also $J = 0$; hence the apices of the S polygon may be located by applying Eqs. (20) to (25), inclusive, or modifications thereof, giving the following results:

$$x_{ab} = 0 = x_{cd}$$
$$y_{ab} = \frac{I_x}{y_a} = y_{cd}$$

$$y_{ac} = 0$$

$$x_{ac} = \frac{I_y}{x_a}$$

$$y_{bd} = 0$$

$$x_{bd} = \frac{I_y}{x_b}$$

Note that x and y for vertical sides correspond to y and x for the horizontal sides.

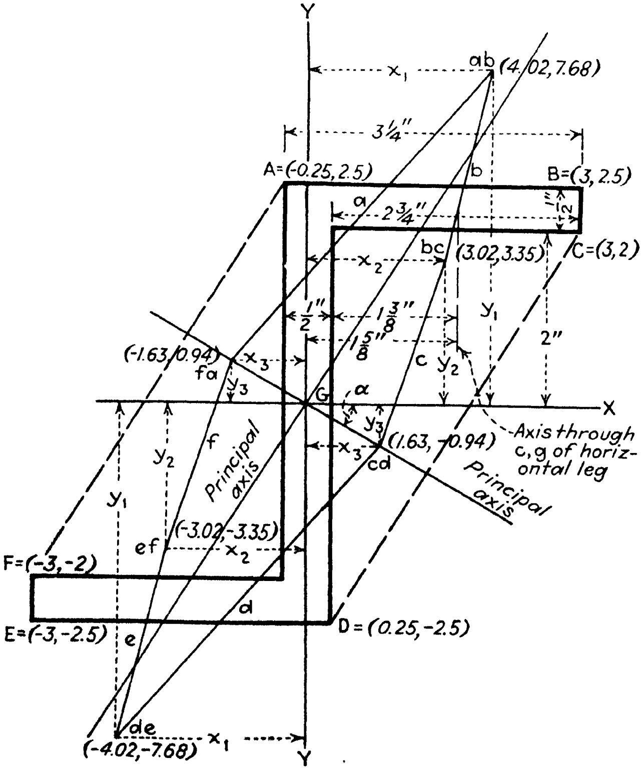


FIG. 10.—*Z* bar; ½ size; *S* polygon: 1 in. = 4 in.^{3}.

Case C. A nonsymmetrical section such as an angle or *Z* bar. For such a case the circumscribing polygon has no axis of symmetry as is clearly shown by Fig. 10 which represents a standard 5 in. by ½ in. *Z* bar with fillets ignored for convenience in computation.

All necessary computations follow.

$$J = J_0 + Akh \qquad J \text{ of web} = 0$$

J for horizontal leg $= (2\frac{3}{4} \times \frac{1}{2}) \times 1\frac{5}{8} \times 2\frac{1}{4} = 5.027$

NOTE.—$A = 2\frac{3}{4} \times \frac{1}{2}$, $k = 1\frac{5}{8}$, $h = 2\frac{1}{4}$.

J for entire Z bar $= 2 \times 5.027 = 10.05$
From handbook $I_x = 19.19$ and $I_y = 9.05$
By Eqs. (20) to (25), inclusive, with proper changes in lettering where necessary, the following values may be obtained:

$$x_{ab} = \frac{10.05}{2.5} = 4.02 \qquad y_{ab} = \frac{19.19}{2.5} = 7.68$$

$$x_{bc} = \frac{9.05}{3} = 3.02 \qquad y_{bc} = \frac{10.05}{3} = 3.35$$

x_{cd} and y_{cd} must be determined from the general formulas (18a) and (19a) with proper substitutions to allow for the rotated axes.

$$x_{cd} = \frac{(3 - 0.25)10.05 - (2 + 2.5)9.05}{-3 \times 2.5 - 0.25 \times 2} = 1.63$$

$$y_{cd} = \frac{(2 + 2.5)(10.05) - (3 - 0.25)(19.19)}{(2)(0.25) - (-2.5)(3)} = -0.94$$

The other apices of the S polygon are evidently symmetrically located on the other sides of the axes from those first computed and need not be figured. The position of the principal axes may be computed by Eq. (7). This gives

$$\tan 2\alpha = \frac{20.10}{9.05 - 19.19} = -1.982$$

hence

$$2\alpha = -(63° - 10')$$
$$\alpha = -(31° - 35')$$
$$\tan \alpha = -0.616$$

35. Equation of Neutral Axis of a Straight Bar.—By placing $f = 0$ in Eq. (9), which is a general equation for the stress at any point m due to a load applied at K, we obtain the following equation in terms of y and x for all points m where $f = 0$, *i.e.*, for the neutral axis for the case of a force applied at K.

$$y = x\frac{Jy_k - I_x x_k}{I_y y_k - Jx_k} - \frac{I_x I_y - J^2}{A(I_y y_k - Jx_k)} \qquad (26)$$

If in the foregoing equation z is substituted for y, q for x, and m for k, the first term, which gives the slope of the neutral axis, is identical with the first term of Eq. (17) for the side of the S polygon corresponding to point m; *i.e.*, the neutral axis for a load applied at m is parallel to the side of the S polygon for point m; the second term differs from the S-polygon equation in having a negative value and having the term A in the denominator; hence, the neutral axis for load at point m lies on the opposite side of the origin from the S-polygon line for this point and at a distance from it equal to the value of S divided by A. If S is expressed in in.^{3} and A in in.^{2}, the distance S/A is in inches.

To obtain the point of intersection for the two neutral axes corresponding to forces applied at two points a and b, respectively, proceed in a manner similar to that used in locating the apices of the S polygon; the results will be shown in Eqs. (27) and (28).

$$x_{ab} = \frac{J(x_b - x_a) + I_y(y_a - y_b)}{A(x_a y_b - y_a x_b)} \tag{27}$$

$$y_{ab} = \frac{J(y_a - y_b) + I_x(x_b - x_a)}{A(x_a y_b - y_a x_b)} \tag{28}$$

The coordinates of the apex thus found equal in value the corresponding coordinates for the S polygon divided by A and are opposite in sign, as should be the case.

If the axes of reference are principal axes, the foregoing equations become

$$x_{ab} = \frac{I_y(y_a - y_b)}{A(x_a y_b - y_a x_b)} \tag{29}$$

$$y_{ab} = \frac{I_x(x_b - x_a)}{A(x_a y_b - y_a x_b)} \tag{30}$$

If the line ab is parallel to an axis of reference, Eqs. (27) and (28) become

ab parallel to XX:

$$y_a = y_b = y, \qquad x_{ab} = -\frac{J}{Ay}, \qquad \text{and} \qquad y_{ab} = -\frac{I_x}{Ay}$$

ab parallel to YY:

$$x_a = x_b = x, \qquad x_{ab} = -\frac{I_y}{Ax}, \qquad \text{and} \qquad y_{ab} = -\frac{J}{Ax}$$

which, if the axes are also axes of symmetry, reduce to the following values:

ab parallel to principal axis *XX*:

$$x_{ab} = 0, \qquad y_{ab} = -\frac{I_x}{Ay}$$

ab parallel to principal axis *YY*:

$$x_{ab} = -\frac{I_y}{Ax}, \qquad y_{ab} = 0$$

The polygon formed by the neutral axes thus obtained by placing the load successively at the various corners of the cross section is called the kernel or kern of the cross section. The kernel for any given cross section may therefore be defined as a space bounded by lines parallel to the sides of the S polygon constructed for the corners of the cross section under consideration. Each boundary line of the kernel is located on the side of the centroid of the cross section opposite to the corresponding parallel side of the S polygon and at a distance from the centroid equal to S/A. The distance from the centroid to the boundary line of the kernel will be in inches if S and A are expressed in inch units.

36. Fibre Stress in a Straight Bar Due to Combined Moment and Direct Stress.—The fibre stress at any point m, Fig. 5, due to force P may now be found as follows:

From Eq. (14)

$$f_m = \frac{N}{A} + \frac{Ng}{S} \qquad (S = \text{section modulus for point } m \text{ and may be plus or minus})$$

Now, let z = distance measured along line OK (Fig. 5) from centroid to the appropriate side of the kernel, and let it be positive if the side of the kernel lies on the opposite side of the centroid from point K or negative if on the same side.

But z (numerically) $= \frac{S}{A}$.

Therefore,

$$f = \frac{N}{A} + \frac{Ng}{Az} = \frac{N}{A}\left(1 + \frac{g}{z}\right) = \frac{N(g + z)}{Az} = \frac{N(g + z)}{S} \qquad (31)$$

In considering the signs to be used for the various terms, consider carefully the contents of Art. 32.

If K is at any point on a boundary line of the kernel, or at any point within the kernel, the stress over the entire cross section will all be of the same character, since for such a case

$z \gtrless g$. Hence, stress at any point on the boundary line of the figure is of the same character and greater than N/A. It follows that if a load is applied anywhere within the kernel, it causes stress of the same character throughout the entire cross section.

The application of Eq. (31) enables one to determine the position of loading for maximum stress at a given point on a cross section of a member subjected both to direct stress and to flexure, such as an arch, since it is only necessary to determine the maximum moment about a kernel point.

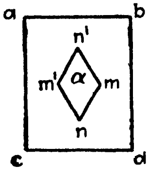


FIG. 11.

For example, if Fig. 11 represents the cross section of an arch, we can determine the loading that will give the maximum fibre stress along the upper fibre *ab* or the lower fibre *cd* by determining the loading that will give a maximum moment about the kernel points *n* and *n'*, respectively.

Application of preceding principle is shown by following examples:

Problem: Determine the location of the kernel points lying in the axis *YY* of a steel arch with the ratio of radius of curvature to depth of rib such that fibre stresses may be computed as if member were a straight bar having the following cross section. Rivet holes may be neglected, since member may be considered to be in compression throughout.
Distance back to back of angles = 45½ in.

Cross section of arch	Area
2 top angles 8 by 6 by 7⁄16 in. with 6-in. leg vertical at 5.93 sq. in. 2 bottom angles 8 by 6 by 7⁄16 in. with 6-in. leg vertical at 5.93 sq. in.	23.72
1 top-cover plate 18 by ½ in. 1 bottom-cover plate 18 by ½ in.	18.00
1 web 45 by ⅜ in	16.87
Total area in square inches	58.59

Solution:

Determination of I:

Cover plates, $2 \times 9 \times (22.75 + 0.25)^2 = 9{,}542$

Flange angles, $4 \times 5.93 \times (22.75 - 1.45)^2 = 10{,}800$

$4 \times 19.3 = 77$

Web, $\frac{1}{12} \cdot \frac{3}{8} \cdot 45^3 = 2{,}840$

Total I in in.^{4} $= 23{,}259$

$$\frac{I}{y} = \frac{23{,}259}{23.25} = 1000 = S\text{ in.}^3$$

and

$$\frac{I}{Ay} = \frac{1000}{58.59} = 17.0\text{ in.} = \text{position of kernel points above and below centroid}$$

Problem: Determine fibre stress at top and bottom of section of arch in previous example if N is a thrust of 400,000 lb. acting at upper kernel point.

Solution:

$$\text{Top} \qquad f = -\frac{400{,}000 \times (17.0 + 17.0)}{1000} = -13{,}600$$

$$\text{Bottom} \qquad f = \frac{400{,}000 \times (17.0 - 17.0)}{1000} = 0$$

The foregoing values are evidently correct since for this case z corresponding to top fibre equals $+g$ and z corresponding to bottom fibre equals $-g$. Hence f at top fibre $= 2N/A$ and f at bottom fibre = zero; that is, stress in one extreme fibre equals twice the average stress and stress in the other extreme fibre equals zero.

Problem: Determine value of maximum tension and compression in cross section of arch of previous example due to a thrust of 200,000 lb. intersecting the cross section at point 40 in. above centroid.

Solution:

$$\text{Top} \qquad f = -\frac{200{,}000}{1000}(40 + 17) = -11{,}400\text{ lb.}$$

$$\text{Bottom} \qquad f = +\frac{200{,}000 \times (40 - 17)}{1000} = 4600\text{ lb.}$$

Preceding values may be checked by applying usual methods, *viz:*

$$f = \frac{P}{A} \pm \frac{My}{I} \qquad \text{but} \qquad \frac{y}{I} = \frac{1}{17.0 \times 58.59} = \frac{1}{1000}$$

whence

$$f = -\frac{200{,}000}{58.59} \pm \frac{200{,}000 \times 40}{58.59 \times 17.0} = -\frac{200{,}000 \times 17.0}{1000} \pm \frac{200{,}000 \times 40}{1000}$$

$$= -\frac{200{,}000 \times 57.0}{1000} \quad \text{or} \quad +\frac{200{,}000}{1000}23.0$$

which equal the expressions previously deduced.

37. Curved Bar. General.—The distribution of normal stress over the cross section of a curved bar differs from that over the cross section of a straight bar owing to the fact that the lengths of fibres of a curved bar lying at equal distances on each side of the central axis as measured between sections perpendicular to that axis are not identical; hence, the unit stresses required to give equal changes in length of these fibres are not identical. This is illustrated by Fig. 12 in which the unit stress in the extreme fibres produced by bending the curved bar shown from the full-line position to the dotted-line position is evidently less in an upper fibre than in a lower fibre, since the upper fibres are longer and hence require less stress to change their lengths the same amount.

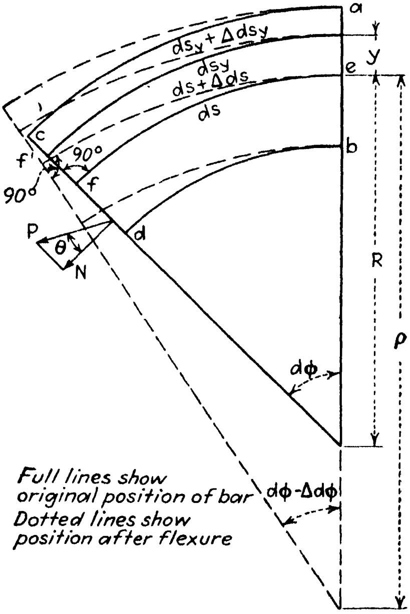


FIG. 12.

In the following discussion, it will be assumed that the central axis of the bar is a curve which lies in a plane containing all the applied forces and an axis of symmetry of every cross section.

38. Distribution of Stress over Cross Section of a Curved Bar. Consider an infinitesimal portion *abcd* of a curved bar as shown in Fig. 12, and assume it to be subjected to a resultant force *P* acting in the longitudinal central plane of the bar (plane of the paper) and making an angle θ with the axis of the bar; also, neglect the distortion due to shear, and make the assumptions used in Art. 29.

Now, let the distance from the central axis to any given fibre of the unloaded bar be y, and let the corresponding distance in the stressed bar be y'.

Let N = normal component of the resultant force P positive when tension.

M = bending moment of force P about a horizontal axis perpendicular to plane of paper and lying at the centre of the cross section, positive when causing a reduction of curvature.

f = fibre stress at any point, positive when tension.

ef = central axis of unstressed bar, and R its radius of curvature.

ef' = position of central axis after application of outer forces, and let ρ be its radius of curvature.

f_y = intensity of stress at distance y from central axis of bar, positive when tension.

ds = length of central axis when unstressed.

$ds + \Delta ds$ = length of central axis after application of force; note that Δds is positive when an increase in ds.

ds_y = length of fibre at distance y from central axis when bar is unstressed.

$ds_y + \Delta ds_y$ = length of same fibre after bar is stressed.

$d\phi$ = central angle subtended by ds.

$d\phi - \Delta d\phi$ = central angle subtended by $ds + \Delta ds$; note that $\Delta d\phi$ is positive when a decrease in $d\phi$.

y and y' = have positive values when above the central axis.

Now,

$$ds = Rd\phi \qquad \text{and} \qquad ds_y = (R + y)d\phi$$

Therefore

$$ds_y = ds + yd\phi$$

also,

$$ds + \Delta ds = \rho(d\phi - \Delta d\phi)$$

hence

$$ds = \rho(d\phi - \Delta d\phi) - \Delta ds$$

Therefore

$$ds_y = \rho(d\phi - \Delta d\phi) - \Delta ds + yd\phi \tag{32}$$

Also,

$$ds_y + \Delta ds_y = (\rho + y')(d\phi - \Delta d\phi) \tag{33}$$

Neglecting the difference between y and y' and substituting in Eq. (33) the value of ds_y from (32), we obtain

$$\Delta ds_y = \Delta ds - y\Delta d\phi \tag{34}$$

Therefore

$$\begin{aligned}\frac{\Delta ds_y}{ds_y} &= \frac{\Delta ds - y\Delta d\phi}{ds_y} = \frac{\Delta ds}{ds_y} - \frac{y\Delta d\phi}{ds_y}\\ &= \frac{ds}{ds}\left(\frac{\Delta ds}{ds_y}\right) - \frac{ds}{ds}\left(\frac{y\Delta d\phi}{ds_y}\right)\\ &= \frac{ds}{ds_y}\left(\frac{\Delta ds}{ds}\right) - \left(\frac{ds}{ds_y}\right)\frac{y\Delta d\phi}{ds}\end{aligned}$$

But

$$\frac{ds}{ds_y} = \frac{R}{R + y}$$

Therefore

$$\frac{\Delta ds_y}{ds_y} = \frac{R}{R + y}\left(\frac{\Delta ds}{ds} - \frac{y\Delta d\phi}{ds}\right) \tag{35}$$

Equation (35) gives the strain in a fibre at a distance y from the central axis, a positive value showing increase in length, *i.e.*, tension in fibre. (Note that for any bar curved as shown in Fig. 12, the value given by Eq. (35) will always be positive in the lower half of the beam where y is negative.) It follows that the stress f_y in the fibre may be found by multiplying the right-hand term of Eq. (35) by E and is as follows, a positive value showing tension:

$$f_y = \frac{ER}{R + y}\left(\frac{\Delta ds}{ds} - \frac{y\Delta d\phi}{ds}\right) \tag{36}$$

Equation (36) is correct whether bar is subjected to an eccentric thrust as indicated or to flexure only.

Now,

$$N = +\int f_v dA = +ER\left(\frac{\Delta ds}{ds}\int\frac{dA}{R+y} - \frac{\Delta d\phi}{ds}\int\frac{ydA}{R+y}\right) \qquad (37)$$

and

$$+M = -\int f_v y dA = -ER\left(\frac{\Delta ds}{ds}\int\frac{ydA}{R+y} - \frac{\Delta d\phi}{ds}\int\frac{y^2dA}{R+y}\right) \qquad (38)$$

But

$$\frac{R}{R+y} = 1 - \frac{y}{R} + \frac{y^2}{R(R+y)} \qquad \text{and} \qquad \frac{Ry}{R+y} = y - \frac{y^2}{R+y}$$

From Eq. (37) we obtain by substitution,

$$\frac{N}{E} = +\frac{\Delta ds}{ds}\left(\int dA - \frac{1}{R}\int ydA + \frac{1}{R}\int\frac{y^2dA}{R+y}\right) - \frac{\Delta d\phi}{ds}\left(\int ydA - \int\frac{y^2dA}{R+y}\right) \qquad (39)$$

For convenience, let $R\int\frac{y^2dA}{R+y} = I_0$. (This may be called a modified moment of inertia.)

Then

$$\int\frac{y^2dA}{R+y} = \frac{I_0}{R}$$

Noting that $\int ydA = 0$ and that $\int dA = A$, we obtain from Eq. (39)

$$\frac{N}{E} = \frac{\Delta ds}{ds}\left(A + \frac{I_0}{R^2}\right) + \frac{\Delta d\phi}{ds}\left(\frac{I_0}{R}\right) \qquad (40)$$

Also noting that $\frac{y}{R+y} = \frac{y}{R} - \frac{y^2}{R(R+y)}$, we obtain from Eq. (38)

$$-\frac{M}{ER} = -\frac{\Delta ds}{ds}\frac{1}{R}\int\frac{y^2dA}{R+y} - \frac{\Delta d\phi}{ds}\int\frac{y^2dA}{R+y}$$
$$= -\frac{\Delta ds}{ds}\frac{I_0}{R^2} - \frac{\Delta d\phi}{ds}\frac{I_0}{R} \qquad (41)$$

Adding Eqs. (40) and (41) and dividing by A gives

$$\frac{\Delta ds}{ds} = \frac{N}{EA} - \frac{M}{EAR} = N_0 \tag{42}$$

equals strain at central axis which is placed equal to N_0 for convenience in future references.

This value when positive indicates increase in length of fibre.

From Eq. (41)

$$\frac{\Delta ds}{ds} = \frac{MR}{EI_0} - R\frac{\Delta d\phi}{ds}$$

Therefore

$$\frac{MR}{EI_0} - R\frac{\Delta d\phi}{ds} = \frac{N}{EA} - \frac{M}{EAR}$$

Therefore

[1]

$$\frac{\Delta d\phi}{ds} = +\frac{M}{EI_0} - \frac{N}{EAR} + \frac{M}{EAR^2} = M_0 \text{ (for convenience)} \tag{43}$$

Substituting in Eq. (36) values of $\Delta ds/ds$ and $\Delta d\phi/ds$ from Eqs. (42) and (43) gives

$$\begin{aligned} f_v &= \frac{ER}{R+y}\left(\frac{N}{EA} - \frac{M}{EAR}\right) - \frac{ERy}{R+y}\left(\frac{M}{EI_0} - \frac{N}{EAR} + \frac{M}{EAR^2}\right) \\ &= \frac{NR}{A(R+y)} - \frac{M}{A(R+y)} - \frac{MRy}{I_0(R+y)} + \frac{Ny}{A(R+y)} - \frac{My}{AR(R+y)} \\ &= \frac{N}{A} - \frac{M}{AR} - \frac{MRy}{(R+y)I_0} \end{aligned} \tag{44}$$

For a straight bar, R = infinity, and Eq. (44) becomes the well-known equation for the case under consideration.

$$f_v = \frac{N}{A} - \frac{My}{I}$$

If bar is subjected to pure flexure only, $N = 0$; hence, Eq. (44) gives the following value for f_v:

---

[1] Note that this equation will not be changed if the sign of Δds in Eq. (34) is negative instead of positive

$$f_y = -\frac{M}{AR} - \frac{MRy}{(R+y)I_0}$$

which for a straight bar becomes

$$f_y = -\frac{My}{I}$$

Owing to the difficulties sometimes met in evaluating I_0, the following nomenclature given by Morley[1] will often be found advantageous:

Let

$$R\int\frac{dA}{R+y} = A' \text{ (This may be called a modified area)}$$

Then

$$\int\frac{y}{R+y}dA = \int\left(1 - \frac{R}{R+y}\right)dA = A - A'$$

and

$$R\int\frac{y^2}{R+y}dA = I_0 = R\int\left(y - R + \frac{R^2}{R+y}\right)dA = R^2(A' - A)$$

since

$$\int y\,dA = 0$$

Substituting foregoing value of I_0 in Eq. (44) gives the following equation:

$$f_y = \frac{N}{A} - \frac{M}{AR} - \frac{My}{R(R+y)(A'-A)} \tag{45}$$

This equation may also be written in the following convenient form:

$$f_y = \frac{N}{A} - \frac{M}{R(A'-A)}\left(\frac{A'}{A} - \frac{R}{R+y}\right) \tag{46}$$

by noting that

$$\frac{M}{R(A'-A)}\left(\frac{y}{R+y}\right) = \frac{M}{R(A'-A)} - \frac{MR}{R(A'-A)(R+y)}$$

---

[1] See reference at end of chapter.

and

$$\frac{M}{R(A' - A)} + \frac{M}{AR} = \frac{MA'}{AR(A' - A)}$$

39. Fibre Stress in a Curved Bar of Rectangular Cross Section.

An analytical solution of the expression $A' = \int \frac{RdA}{R + y}$ for a bar of *rectangular* section may be readily made as follows:

Let the bar have a width of b and a depth of d.

Then

$$\int_{-\frac{d}{2}}^{\frac{d}{2}} \frac{RdA}{R + y} = Rb \int_{-\frac{d}{2}}^{\frac{d}{2}} \frac{dy}{R + y} = Rb \log_e \frac{2R + d}{2R - d}$$

This may be solved directly by the use of Napierian logarithms or may be evaluated by Taylor's theorem[1] at shown in the footnote:

Therefore

$$A' = Rb \log_e \frac{2R + d}{2R - d} = \\ bd\left[1 + \frac{d^2}{12R^2} + \frac{d^4}{80R^4} + \frac{d^6}{448R^6} + \cdots\right]$$

---

[1] $\log_e \frac{2R + d}{2R - d} = \log_e (2R + d) - \log_e (2R - d)$

Let $2R = x$; then by Taylor's theorem,

$$\log_e (x + d) = \log x + \frac{d}{x} - \frac{d^2}{2x^2} + \frac{d^3}{3x^3} - \frac{d^4}{4x^4} + \frac{d^5}{5x^5} - \frac{d^6}{6x^6} + \frac{d^7}{7x^7}, \text{ etc.}$$

and

$$\log_e (x - d) = \log x - \frac{d}{x} - \frac{d^2}{2x^2} - \frac{d^3}{3x^3} - \frac{d^4}{4x^4} - \frac{d^5}{5x^5} - \frac{d^6}{6x^6} - \frac{d^7}{7x^7}, \text{ etc.}$$

Therefore

$$\log_e (x + d) - \log_e (x - d) = \frac{2d}{x} + \frac{2d^3}{3x^3} + \frac{2d^5}{5x^5} + \frac{2d^7}{7x^7} + \ldots$$

and

$$\log_e (2R + d) - \log_e (2R - d) = \frac{d}{R} + \frac{d^3}{12R^3} + \frac{d^5}{80R^5} + \frac{d^7}{448R^7} + \ldots$$

Therefore

$$A' - A = bd\left[\frac{d^2}{12R^2} + \frac{d^4}{80R^4} + \frac{d^6}{448R^6} + \cdots\right] \tag{47}$$

Since $I_0 = R^2(A' - A)$, its value for a rectangle equals

$$bd\left[\frac{d^2}{12} + \frac{d^4}{80R^2} + \frac{d^6}{448R^4} + \cdots\right]$$

The third and all following terms are so small in comparison with the first term that practically no error is made in omitting them; hence we may write for a rectangular section:

$$\begin{aligned} I_0 &= bd\left(\frac{d^2}{12} + \frac{d^4}{80R^2}\right) \\ &= \frac{bd^3}{12}\left(1 + \frac{3d^2}{20R^2}\right) = I\left(1 + \frac{3d^2}{20R^2}\right) \end{aligned} \tag{48}$$

In general, in structural design, it may be said that an approximation of 1 per cent in the computations may be considered permissible; hence for a bar of *rectangular* section I_0 may be considered as equal to I when $\frac{3}{20}\frac{d^2}{R^2} \leqq 0.01$, *i.e.*, when

$$\frac{d^2}{R^2} \leqq \frac{1}{15} \qquad \text{or} \qquad \frac{d}{R} \leqq \frac{1}{3.87} \text{ or, say, } \frac{1}{4}$$

The ratio between the two last terms of Eq. (44) for rectangular bars where I_0 may be taken as equal to I is as follows:

$$\frac{M}{AR} \div \frac{MRy}{(R + y)I} = \frac{(R + y)I}{AR^2y}$$

which for a rectangular bar equals

$$\frac{(R + y)\frac{1}{12}bd^3}{bdR^2y} = \frac{d^2(R + y)}{12R^2y}$$

For maximum fibre stress $y = d/2$; hence for this case the ratio between these terms equals

$$\frac{d^2\left(R + \frac{d}{2}\right)}{12R^2\frac{d}{2}} = \frac{d(2R + d)}{12R^2}$$

Let $d = cR$ in which c is a constant.
Then the foregoing ratio equals

$$\frac{cR(2R + cR)}{12R^2} = \frac{2c + c^2}{12}$$

When $c = 0.0583$,* this ratio $= 1/100$

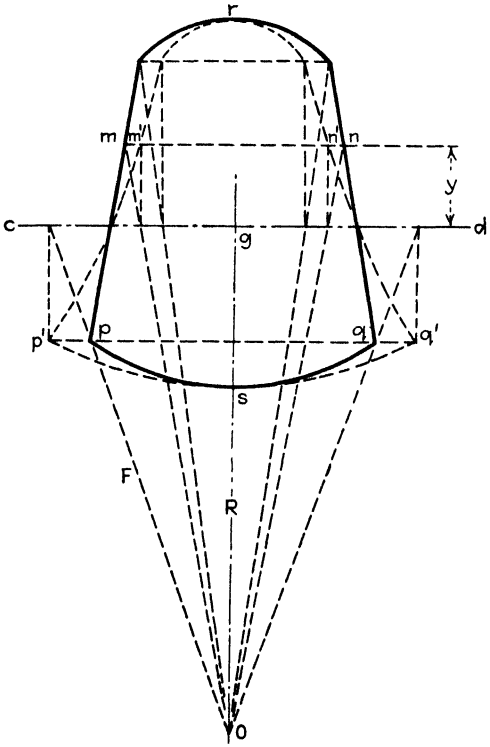


FIG. 13.

In other words, for curved bars of rectangular section in which $d/R = 0.0583$, or say $6/100$, the maximum flexure stress as determined by the last two terms of Eq. (44) will be in error not more than 1 per cent if the second term of the equation is neglected and if I_0 is taken as equal to I. If the direct stress is of the same character as the stress due to flexure, the error in the combined stress will be less than 1 per cent; if the direct stress is of the opposite character to that due to flexure, the error in the combined stress will be larger than 1 per cent and may reach a high percentage, although the actual stress may be small. Such a condition is, however, seldom the controlling factor in design.

---

* For this value of c, $I_0 = I(1 + 1/2000)$.

An I-beam section can be considered as composed of rectangular parts; hence, conclusions apply to such a section also.

40. Fibre Stress in Bars of Irregular Section.—For such bars, Eq. (45) may be most advantageously applied, the modified area A' being determined graphically as follows: Referring to Fig. 13,

Let the cross section of the bar be *mrnqsp*, the centroid being at g.

Let the centre of curvature be at O and the radius of curvature for the layer passing through g equal R.

Let w be the width of the cross section at any point distant y from the axis passing through g.

Then

$$A' = \int \frac{RdA}{R+y} = \int \frac{Rwdy}{R+y}$$

The modified length $\frac{Rw}{R+y}$ of any line such as *mn* may be determined as follows:

Draw the lines *mO* and *nO*, and from the points where these lines cut the axis *cd* erect perpendiculars intersecting the line *mn* at m' and n'.

Evidently

$$\frac{m'n'}{mn} = \frac{R}{R+y}$$

Therefore

$$m'n' = mn\frac{R}{R+y} = \frac{Rw}{R+y}$$

In this manner the modified width may be determined at as many points as are necessary to locate the perimeter of the modified cross section $m'rn'q'sp'$, the area of which will be the modified area.

The modified area may be determined with considerable accuracy by Simpson's rule for integration provided the figure is constructed with great care.

41. Illustrative Problems.—The following examples clearly illustrate the application of the foregoing method:

Problem: Compute the value of the maximum tension and compression in a rectangular bar 2 in. wide and 3 in. deep with its longitudinal axis curved in a plane parallel to its depth, the mean radius of curvature being 4 in. The bar is subjected to a couple of 30,000 in.-lb. tending to reduce its curvature, *i.e.*, to produce compression in the upper fibre; it has, however, no direct thrust at any section.

Solution: From Eq. (47) $A' = 6(1 + 0.04687 + 0.00396) = 6.30498$
Inserting this value in Eq. (46) gives

$$f_v = 0 - \frac{30,000}{4(0.305)}\left(\frac{6.305}{6.000} - \frac{4}{5.5}\right) = -7990 \text{ (maximum compression)}$$

For the maximum tension the formula becomes

$$f_v = -\frac{30,000}{4(0.305)}\left(\frac{6.305}{6.000} - \frac{4}{2.5}\right) = +13,500 \text{ (maximum tension in lower fibre)}$$

By the ordinary formula for a straight beam, these stresses would each be 10,000 lb.

The preceding example well illustrates the difference between the fibre stresses in a straight bar and those in a curved bar the radius of which is very small compared with its thickness. For ordinary bridge arches, the difference between the two methods will be hardly appreciable, as is indicated by the investigation of Art. 39. For sewers, subway roofs, and similar arched structures the application of the more exact formulas may be sometimes desirable.

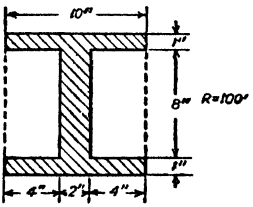


FIG. 14.

Problem: Compute I_0 for the cross section shown crosshatched in Fig. 14.

Solution:
For circumscribing rectangle,

$$I_0 = \frac{1}{12}10 \times 10 \times 10 \times 10\left[1 + \frac{3 \times 100}{20(100 \times 12)(100 \times 12)}\right]$$
$$= \frac{10,000}{12}\left(1 + \frac{1}{96,000}\right)$$

For nonshaded portions,

$$I_0 = 2 \times \frac{1}{12} \times 8 \times 8 \times 8 \times 8\left[1 + \frac{3 \times 8 \times 8}{20(100 \times 12)(100 \times 12)}\right]$$
$$= \frac{2048}{3}\left(1 + \frac{1}{150,000}\right)$$

Then, for the crosshatched portion,

$$\begin{aligned}
I_0 &= 833\tfrac{1}{3}\left(1 + \frac{1}{96{,}000}\right) - 682\tfrac{2}{3}\left(1 + \frac{1}{150{,}000}\right) \\
&= 833\tfrac{1}{3} - 682\tfrac{2}{3} + \frac{833\tfrac{1}{3}}{96{,}000} - \frac{682\tfrac{2}{3}}{150{,}000} \\
&= 833\tfrac{1}{3} - 682\tfrac{2}{3} + \frac{125{,}000{,}000 - 65{,}536{,}000}{150{,}000 \times 96{,}000} \\
&= 833\tfrac{1}{3} - 682\tfrac{2}{3} + \frac{59{,}000{,}000}{150{,}000 \times 96{,}000} \\
&= 833\tfrac{1}{3} - 682\tfrac{2}{3} + \frac{59}{150 \times 96} \\
&= 833.337 - 682.667 = 150.670 \text{ in.}^4
\end{aligned}$$

Problems

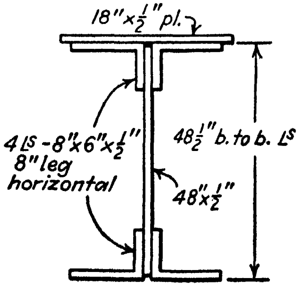
1. Determine the stress per square inch at points A, B, C, D, E, and F of the Z bar shown in Fig. 10, due to an axial tensile force of 10,000 lb. applied at 2 in. to left and ¼ in. below point A. State both the magnitude and the character of the stress.

2. Plot the S polygon and the kernel of the cross section for a 5- by 3½- by ½-in. angle. Using these, compute the maximum tension and compression due to an axial tension of 20,000 lb. acting at:

(a) 2½ in. from the back of the angle in the centre of the 5-in. leg.

(b) 2 in. from the back of the angle in the centre of the 3½-in. leg.

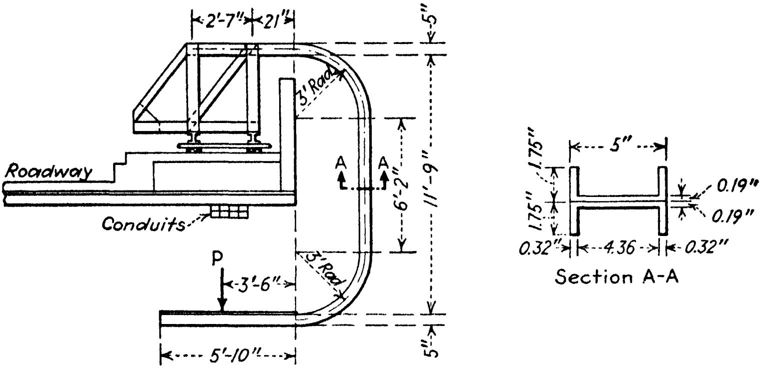
3. Determine position of kernel points for the plate-girder arch rib shown in the figure and using the values thus found determine maximum tension and compression due to an axial thrust of 100,000 lb. applied 10 in. above web centre. Centre of gravity of angles 1.5 in. from back of 8-in. legs. I of angle about axis parallel to 8-in. leg = 21.7 in.^{4}. A of one angle = 6.75 sq. in.



4. *a.* Determine for the member given in the illustrative problem in Art. 36 the distance above the centre of the girder that a thrust of 200,000 lb., applied in a vertical plane passing through the centre of the girder, may act without causing a compressive fibre stress in excess of 20,000 lb. per square inch in the upper fibre of the member.

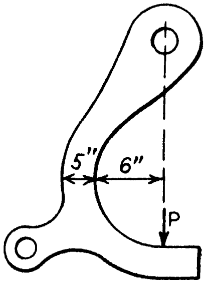
b. Determine the character and magnitude of the fibre stress in lower fibre of member for the position of the thrust computed in *a*.

5. The structure shown in the figure at top of page 100 is a hook designed to carry a platform for workmen engaged in laying telephone conduits under a bridge floor. The hook is made out of two aluminum channels back to back as shown. If the allowable fibre stress is 15,000 lb. per square inch, determine the allowable value of the load P, acting as shown.



6. Compute I_0 for arch having a cross section identical with that in Art. 36, with $R = 60$ ft. Results to be precise to three places of decimals.

7. The structure shown in this figure is a hook for locking the end of a bascule bridge. Determine the force P required to develop a maximum fibre stress of one-half the yield point of the material. The material is cast steel annealed, having a yield point of 38,000 lb. per square inch. The hook is 6 in. thick.



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CHAPTER IV

DEFLECTION OF A CURVED BAR

42. Movement of Points on Axis Due to Changes in Length and Curvature.—Consider a curved bar of homogeneous material the original axis of which is shown by $BB'C$ (Fig. 1), and determine the movement of point C with respect to point B due to each of the following causes:

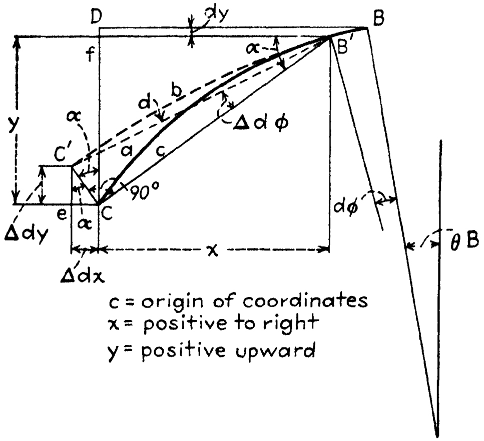

Fig. 1.—Properties of curved bars.

1. Change in axial length of bar between B and C.
2. Change in curvature of bar between B and C.
3. Change in position and slope of bar resulting from movement of point B or, if B is at a support, from change in position and slope of the support.

Let Δx_c = total horizontal movement of point C, positive to right.
Δx_B = total horizontal movement of point B, positive to right.
Δy_c = total vertical movement of point C, positive upward.
$\Delta x'$ = horizontal movement of point C due to change in length of bar.

$\Delta y'$ = vertical movement of point C due to change in length of bar.

$\Delta x''$ = horizontal movement of point C due to change in curvature of bar between B and C.

$\Delta y''$ = vertical movement of point C due to change in curvature of bar between B and C.

Δy_B = vertical movement of point B, positive upward.

$\Delta \phi_B$ = angular movement of point B, positive when causing decrease in ϕ_B.

$\Delta x'''$ = horizontal movement of point C due to horizontal and angular movements of point B.

$\Delta y'''$ = vertical movement of point C due to vertical and angular movements of point B.

S = length of axis of bar between any two sections.

Δds = change in length of the infinitesimal original length ds of portion of bar indicated by BB'.

Let origin of coordinates be at original position of point C, and consider x positive to right, y positive upward, and ϕ positive as shown.

Evidently the horizontal and vertical movements of point C due to change in length Δds of portion BB' of bar equal the horizontal and vertical movements of B' due to same cause and are given by the following expressions:

$$\Delta ds\frac{dx}{ds} \quad \text{and} \quad \Delta ds\frac{dy}{ds}$$

Therefore the total movements of this point due to increase in length of entire bar are, respectively,

$$\Delta x' = -\int \Delta ds\frac{dx}{ds} \quad \text{and} \quad \Delta y' = -\int \Delta ds\frac{dy}{ds}$$

If $\Delta ds/ds$ is constant, as would be the case for a constant unit stress in the bar, it would equal f/E. Therefore, for this case,

$$\Delta x' = -\frac{f}{E}x \quad \text{and} \quad \Delta y' = -\frac{f}{E}y$$

whence, $\Delta x'/\Delta y' = x/y$, hence point c would move on line $\overset{?}{B}'C$.

Now, let $B'aC$ = original position of portion of bar to left of B'.

$B'bC'$ = position of foregoing portion of bar after decrease $\Delta d\phi$ in curvature between B and B' if the infinitesimal movement of point B' upward and to the left is neglected.

$B'cC$ = original position of chord of arc $B'aC$.
$B'dC'$ = position of same chord after a decrease $\Delta d\phi$ in curvature.

The angular movement of chord $B'cC$ due to change $\Delta d\phi$ in curvature of angle $d\phi$ equals $\Delta d\phi$; hence, the angle at B' between chords $B'cC$ and $B'dC'$ equals $\Delta d\phi$. Moreover, the arc CC' through which the point C moves because of angular change in BB' is essentially equal to the normal CC' to the chord $B'cC$. Therefore, since $\Delta d\phi$ is infinitesimal, the normal

$$CC' = \text{chord } B'cC(\Delta d\phi)$$

hence,

$$\Delta d\phi = \frac{CC'}{B'cC}$$

But

$$\angle CC'e = \angle CcB'f = \alpha$$

Therefore

$$\frac{Ce}{CC'} = \frac{Cf}{B'cC} = \sin\alpha, \qquad \text{but} \qquad \frac{CC'}{B'cC} = \Delta d\phi$$

hence

$$Ce = -\Delta dx = Cf\Delta d\phi$$

Therefore $\Delta dx = -y\Delta d\phi$, the negative sign showing movement to left for curvature resulting from positive moment in bar.

Proceeding in a similar manner, we obtain the following expression:

$$eC' = \Delta dy = x\Delta d\phi$$

the positive sign showing upward movement for positive bending moment.

Integration of the foregoing general expressions gives the following values for the total movement of point C due to change in curvature only throughout the entire length of bar BC:

$$\Delta x'' = -\int^{S} y\Delta d\phi\frac{ds}{ds}$$

Similarly,

$$\Delta y'' = \int^{S} x\Delta d\phi\frac{ds}{ds}$$

In deriving the preceding equations, the infinitesimal movements of point B' both upward and to the left, which have been neglected, are infinitesimals of the second order as compared with the movements of point C, and hence the error in the final results due to neglecting them is infinitesimal; also, the other approximations made are infinitesimal in their effect upon the final result.

In the foregoing equations $\Delta d\phi$ is considered positive when a decrease, since it thereby corresponds to positive moment in the bar.

To determine the horizontal and vertical deflections of point C due to angular rotation at point B, the foregoing method may also be used. If angular movement is $\Delta\phi_B$ and is a decrease, *i.e.*, if ϕ_B is decreased in size, the total movement of point C due to angular combined with horizontal and vertical movements of point B is given by the following expressions:

$$\Delta x'\,'' = \Delta x_B - CD\Delta\phi_B$$
$$\Delta y'\,'' = \Delta y_B + DB\Delta\phi_B$$

It follows that

$$\Delta x_c = \Delta x' + \Delta x'' + \Delta x'\,'' = \int_0^{x=DB}\left(-\frac{\Delta ds}{ds}\right)dx - \int_0^S y\Delta d\phi\frac{ds}{ds} + \Delta x_B - CD\Delta\phi_B$$

$$\Delta y_c = \Delta y' + \Delta y'' + \Delta y'\,'' = \int_0^{y=CD}\left(-\frac{\Delta ds}{ds}\right)(dy) + \int_0^S x\Delta d\phi\frac{ds}{ds} + \Delta y_B + DB\Delta\phi_B$$

From previous equations in Art. 38

$$\frac{\Delta ds}{ds} = N_0 \qquad \text{and} \qquad \frac{\Delta d\phi}{ds} = M_0$$

Therefore

$$\Delta x_c = \int_0^{DB}(-N_0)dx - \int_0^S M_0 y ds + \Delta x_B - CD\Delta\phi_B \qquad (1)$$

$$\Delta y_c = \int_0^{CD}(-N_0)dy + \int_0^S M_0 x ds + \Delta y_B + DB\Delta\phi_B \qquad (2)$$

Also, since $\Delta d\phi$ = change in magnitude of the angle $d\phi$ due to change in curvature between B and B', the total change $\Delta\phi$ in curvature between B and $C = \int\Delta d\phi$

Therefore

$$\Delta\phi = \int_0^S \frac{\Delta d\phi}{ds} ds = + \int_0^S M_0 ds \qquad (3)$$

In the preceding equations

$$N_0 = \frac{N}{EA} - \frac{M}{EAR} \qquad (4)$$

and

$$M_0 = \frac{M}{EI_0} - \frac{N}{EAR} + \frac{M}{EAR^2} \qquad (5)$$

In the foregoing equations N equals normal component of resultant stress on section under consideration and is positive when tension and M equals bending moment at same section and is positive when causing compression in upper fibre.

43. Test of Equations.—Equations (1) and (2) may be tested by applying them to the case of an arch of infinite radius, *i.e.*, a straight bar, for which case

$$N_0 = \frac{N}{EA} \quad \text{and} \quad M_0 = \frac{M}{EI}$$

For the case of a horizontal bar of length L carrying axial stress only, Δx = horizontal movement to the right of the left end of the bar and is given by Eq. (1) and is as follows for a pull of N applied along the central axis provided the right end is fixed in position.

$$\Delta x = -\int_0^L \frac{N}{EA} dx = -\frac{NL}{EA}$$

The foregoing expression is evidently correct in both magnitude and direction.

For the case of a horizontal bar of length L carrying both flexure and direct stress and fixed in direction and in position at the centre, the vertical movement of the left end with respect to centre is given by Eq. (2) and is as follows:

$$\Delta y = \int_0^{\frac{L}{2}} \frac{Mxdx}{EI}$$

The preceding equation is evidently correct, since the result is identical with that given by the deflection formula

$$\delta = \int \frac{Mmdx}{EI}$$

The reader should observe that if the axial thrust in a straight beam is considerable as compared with the bending moment, the beam would have some vertical deflection due to buckling such as occurs in columns, and hence Eq. (2) would not give the exact vertical deflection for such a case of combined thrust and bending.

44. Application of Equations to Two-hinged Arch Ribs. *Case A.*—Two-hinged arch, span = L, abutments not fixed, supports at same level. Temperature and shrinkage considered. Ratio of depth to radius of curvature small.

This structure has four unknown reaction components; hence, only one equation in addition to those of statics is required for its solution. Equation (1) furnishes such an equation.

Let S = length of arch axis.

ΔL = increase in horizontal distance between ends of arch due to movement of abutments under load or to any other cause equals $(\Delta x_B - \Delta x_c)$.

ϵ = coefficient of expansion.

ϵ' = coefficient of shrinkage.[1]

t = change of temperature in degrees (+ for increase).

Then from Eq. (1), by substituting N/EA for N_0 and M/EI for M_0 and noting that $CD = 0$, since hinges are level, and including terms for temperature and shrinkage, we obtain the following expression for ΔL:

$$-\Delta L = -\int_0^L \frac{Ndx}{EA} - \int_0^S \frac{My}{EI}ds - \epsilon tL + \epsilon' L \qquad (6)$$

Let M_x = moment at any section of a simple end-supported straight beam of span L and lying parallel to line joining end hinges.

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[1] Note that temperature and shrinkage changes may be combined. The German Committee for Reinforced Concrete specifies that shrinkage for framed structures and for arches and vaults with total reinforcement = 0.5 per cent or more should be taken as equivalent to a temperature drop of 15°C. In the latter types of structures if reinforcement is less than 0.5 per cent, consider shrinkage equivalent to a temperature drop of 20°C.

H = component of reactions at either end of arch acting parallel to line joining end hinges and assumed positive when outward corresponding to tension in arch.

We may then write:

$$M = M_x + Hy$$

Substituting this in Eq. (6) gives

$$-\Delta L + \epsilon tL - \epsilon' L = -\int_0^L \frac{Ndx}{EA} - \int_0^S \frac{M_x y}{EI} ds - H\int_0^S \frac{y^2 ds}{EI} \qquad (7)$$

whence

$$H = \frac{-\int_0^L \frac{Ndx}{EA} - \int_0^S \frac{M_x yds}{EI} + \Delta L - \epsilon tL + \epsilon' L}{\int_0^S \frac{y^2 ds}{EI}} \qquad (8)$$

If foregoing expression proves to be negative, as would generally be the case, it would show that the horizontal reactions act toward centre of span.

If the arch rib is rectangular of depth d and width b, we may substitute for I its value $\frac{1}{12}bd^3$. If the arch is also assumed to be divided into n equal parts of finite but small length Δs, with horizontal projection $= \Delta x$, and to be of homogeneous material, Eq. (8) may be written as follows:

$$H = -\frac{\Sigma N\Delta x/d}{12\sum \frac{y^2}{d^3}\Delta s} - \frac{\Sigma M_x y}{\Sigma y^2} + \frac{Eb\Delta L}{12\sum \frac{y^2}{d^3}\Delta s} - \frac{E\epsilon btL}{12\Delta s\sum \frac{y^2}{d^3}} + \frac{\epsilon' bLE}{12\Delta s\sum \frac{y^2}{d^3}} \qquad (9)$$

These terms represent, respectively, the influence upon the horizontal thrust of the following factors:

Term 1. Deformation of the arch rib due to axial stress, assumed in equation as tension.

Term 2. Bending of the arch rib.

Term 3. Longitudinal displacement of the abutments due to any cause.

Term 4. Changes of length of arch rib due to temperature.

Term 5. Shortening of arch rib due to shrinkage.

The first term in Eq. (9) is evidently a function of the horizontal force and cannot therefore be directly computed from the equation; it is common to omit this term in preliminary computations and correct for it later if necessary. The third term is difficult to estimate; for rigid abutments, its value is zero; for other cases, it may be assumed zero in preliminary computations and revised later if warranted by the conditions. The value of the second term can evidently be obtained for any two-hinged arch by dividing the arch into segments sufficiently close together to make negligible the error of using for M_x the average moment upon any particular section and of using for y and d the ordinate and depth of the centre of the section. For arches of uniform cross section and of regular curves, such as parabolas and circles, such a procedure is unnecessary, since analytical expressions for y and M can be developed by other methods, and the terms integrated as shown in Chaps. V and VI.

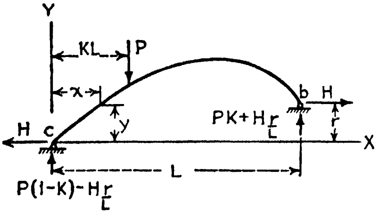

FIG. 2.

Case B.—Two-hinged arch. Supports on different levels.[1] Temperature and shrinkage considered. Ratio of depth to radius of curvature small.

Let N = normal component of resultant stress on any section of arch, positive when tension.

M = bending moment at any section of arch, positive when causing a reduction of curvature.

H = horizontal component of reaction at either end of arch assumed positive when acting outward.

M_p = bending moment at any section of arch due to applied load P assuming end b on rollers and H removed.

A_c = cross-sectional area of the arch at the point where the tangent to the arch axis is horizontal.

ϵ = temperature coefficient.

t = change of temperature in degrees.

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[1] This method was developed by Prof. E. Mirabelli, assistant professor of structural design, Massachusetts Institute of Technology.

ΔL = increase in horizontal distance between ends of arch due to horizontal movement of abutments under load or to any other cause $= +(\Delta x_b - \Delta x_c)$.

Δr = increase in vertical distance between ends of arch due to vertical movement of abutments under load or to any other cause $= +(\Delta y_b - \Delta y_c)$.

Applying Eqs. (1) and (2) in Art. 42 to the arch shown in Fig. 2 and allowing for temperature and shrinkage gives the following expressions if temperature and shrinkage-coefficients are combined:

$$\Delta x_c = -\int \frac{N}{EA}dx - \int \frac{M}{EI}yds + \Delta x_b - r\Delta\phi_b - \int \epsilon t dx \qquad (10)$$

$$\Delta y_c = -\int \frac{N}{EA}dy + \int \frac{M}{EI}xds + \Delta y_b + L\Delta\phi_b - \int \epsilon t dy \qquad (11)$$

Eqs. (10) and (11) may be rewritten as follows:

$$\Delta\phi_b = \frac{1}{r}\left[-\int \frac{Ndx}{EA} - \int \frac{Myds}{EI} + \Delta L - \int \epsilon t dx\right] \qquad (12)$$

$$\Delta\phi_b = \frac{1}{L}\left[+\int \frac{Ndy}{EA} - \int \frac{Mxds}{EI} - \Delta r + \int \epsilon t dy\right] \qquad (13)$$

If the expressions for $\Delta\phi_b$ in Eqs. (12) and (13) are equated, the following equation is obtained:

$$\Delta L = \int \frac{N}{EA}\left(dx + \frac{r}{L}dy\right) + \int \frac{M}{EI}\left(y - \frac{r}{L}x\right)ds - \frac{r}{L}\Delta r + \epsilon t \int \left(dx + \frac{r}{L}dy\right) \qquad (14)$$

But

$$M = M_p - \frac{Hr}{L}x + Hy$$

Also, it may be assumed that

$$\frac{N}{A} = \frac{H}{A}$$

If these expressions for M and N/A are substituted in Eq. (14) and integrated between proper limits, the following expression for H is obtained:

$$H = \frac{-\int M_v\left(y - \frac{r}{L}x\right)\frac{ds}{EI} - \epsilon tL\left(1 + \frac{r^2}{L^2}\right) + \Delta L + \frac{r}{L}\Delta r}{\frac{L}{EA_c}\left(1 + \frac{r^2}{L^2}\right) + \int\left(y - \frac{r}{L}x\right)^2\frac{ds}{EI}} \qquad (15)$$

Equation (8) developed for case A may be obtained by letting r of Eq. (15) equal zero, placing $N = H$, and adding term for shrinkage.

If the arch axis is a curve involving difficult integration, or if the arch has a variable moment of inertia, the integral signs may be replaced by summation signs, and ds by Δs, and Eq. (15) evaluated in the same manner as in the case of the fixed-ended arch of variable cross section which is considered in Art. 45, also in Chap. VIII. The evaluation of Eq. (15) may be facilitated if it is noted that the quantity $\left(y - \frac{r}{L}x\right)$ is the vertical distance between a point on the axis of the arch and the line joining the two end hinges and may be easily scaled on a layout of the arch.

Case C.—Two-hinged arch span $= L$, rigid abutments, supports at same elevation, temperature changes and shrinkage neglected. Ratio of depth to radius of curvature small.

For this case also Eq. (1) gives the equation necessary for solution in addition to the equations of statics and is as follows:

$$\Delta x_0 = 0 = -\int_0^L N_0 dx - \int_0^S M_0 y ds \qquad (16)$$

Also,

$$N_0 = \frac{N}{EA} \quad \text{and} \quad M_0 = \frac{M}{EI}$$

hence resulting equation becomes

$$0 = -\int_0^L \frac{N}{EA}dx - \int_0^S \frac{Myds}{EI} \qquad (17)$$

which may be expanded in the same manner as Eq. (6) of case A.

45. No-hinged Arch of Homogeneous Material and with Variable Section.—No-hinged arch, rigid abutments, span $= L$, supports at equal elevation. Ratio of depth to radius of curvature small. Temperature changes and shrinkage neglected.

Such a structure has six unknown reaction equations; hence, three equations are necessary for solution in addition to the three equations of statics.

Equations (1), (2), and (3) furnish the necessary equations to determine the change in position of one end with respect to the other, the following terms being equal to zero:

Δx_c, Δy_c, Δx_B, Δy_B, $CD\Delta\phi_B$, $DB\Delta\phi_B$, $\Delta\phi$[*]

The resulting equations for this case will, therefore, be as follows:

$$\Delta x_c = -\int_0^L N_0 dx - \int_0^S M_0 y ds = 0 \tag{18}$$

which when plus equals deflection of C to right

$$\Delta y_c = -\int_0^{CD} N_0 dy + \int_0^S M_0 x ds = 0 \tag{19}$$

$$\Delta\phi = +\int_0^S M_0 ds = 0 \tag{20}$$

If the radius of curvature is large compared with the depth of the arch rib, these equations become

$$\Delta x_c = -\int_0^L \frac{N dx}{EA} - \int_0^S \frac{My}{EI} ds^{\dagger} = 0 \tag{21}$$

$$\Delta y_c = -\int_0^y \frac{N dy}{EA} + \int_0^S \frac{Mxds^{\dagger}}{EI} = 0 \tag{22}$$

$$\Delta\phi = \int_0^S \frac{Mds^{\dagger}}{EI} = 0 \tag{23}$$

For the case of arch bridges, the value of the term containing N is usually small as compared with the other terms and is often neglected in preliminary computations, the result being corrected later to allow for that term. If this term is omitted, the following expressions result:

$$-\int_0^S \frac{Myds}{EI} = \Delta x = 0 \tag{24}$$

$$\int_0^S \frac{Mxds}{EI} = \Delta y = 0 \tag{25}$$

$$\int_0^S \frac{Mds}{EI} = \Delta\phi = 0 \tag{26}$$

---

[*] Rigid abutments imply no change in direction of springing lines; hence $\Delta\phi_B$ and $\Delta\phi$ each equal zero.

[†] Note that these terms $= \int \frac{Mmdx}{EI}$ where m = moment due to unit force in first two cases and to unit moment in last case.

Equations (24), (25), and (26) may be used in determining arch reactions, as illustrated in the following discussion.

Since in the case of the no-hinged arch, the cross section is usually variable, and the axis an irregular curve; the integrations appearing in Eqs. (24), (25), and (26) must be changed to summations, the arch being divided into a series of short sections each being considered separately. The summations may be advantageously performed by dividing the arch at the crown into halves and properly combining the crown deflection for each cantilever portion of the divided arch.

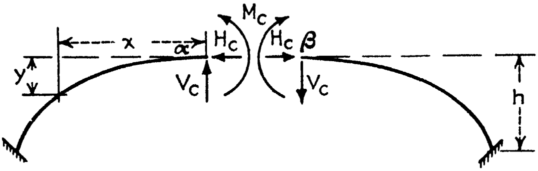


FIG. 3.—No-hinged arch of homogeneous material and with variable section.

Let the arch be divided as shown in Fig. 3, and let M_c, H_c, and V_c be the forces acting at the crown of each section of the arch.

Let origin of coordinates be at crown and let x be positive both to left and to right and y positive downward, moment causing compression in upper fibre being considered positive.

Let δ_y = vertical deflection of point α down.

δ'_y = vertical deflection of point β down.

δ_x = horizontal deflection of point α left.

δ'_x = horizontal deflection of point β left.

$\Delta\phi$ = change in central angle of left half of arch, positive if decrease.

$\Delta\phi'$ = change in central angle of right half of arch, positive if decrease.

The following equations may now be written:

$$\delta_y = \delta'_y; \qquad \delta_x = \delta'_x; \qquad \text{and} \qquad \Delta\phi = -\Delta\phi'$$

The values of the various terms in foregoing equations may be obtained from Eqs. (24), (25), and (26) by expressing the integral terms as summations and by referring to origin of coordinates at crown instead of springing, resulting in the following terms in which M_L and M_R refer, respectively, to moment on left and on right of the crown and have the values given on page 113.

$$\delta_y = -\sum \frac{M_L x \Delta s}{EI} \qquad \delta_x = \sum \frac{M_L y \Delta s}{EI} \qquad \Delta\phi = \sum \frac{M_L \Delta s}{EI}$$

$$\delta'_y = -\sum \frac{M_R x \Delta s}{EI} \qquad \delta'_x = -\sum \frac{M_R y \Delta s}{EI}^{*} \qquad \Delta\phi' = \sum \frac{M_R \Delta s}{EI}$$

We may now write the following equations:

$$\sum \frac{M_R x \Delta s}{EI} - \sum \frac{M_L x \Delta s}{EI} = 0 \tag{27}$$

$$\sum \frac{M_L y \Delta s}{EI} + \sum \frac{M_R y \Delta s}{EI} = 0 \tag{28}$$

$$\sum \frac{M_L \Delta s}{EI} + \sum \frac{M_R \Delta s}{EI} = 0 \tag{29}$$

If arch is divided into sections so that $\Delta s/I$ is constant, the preceding equations become, for an arch of homogeneous material, as follows:

$$\Sigma M_R x - \Sigma M_L x = 0 \tag{30}$$

$$\Sigma M_L y + \Sigma M_R y = 0 \tag{31}$$

$$\Sigma M_L + \Sigma M_R = 0 \tag{32}$$

If m_L = moment of applied forces about point x of left cantilever, and m_R equals corresponding moment on right cantilever, the following expressions may be written for M_R and M_L:

$$M_R = m_R + H_c y - V_c x + M_c$$
$$M_L = m_L + H_c y + V_c x + M_c$$

Note that m_L and m_R will, in general, have negative values, since loads, in general, act downward.

Inserting preceding values in Eqs. (30) to (32) gives the following expressions:

From Eq. (30),

$$\Sigma m_R x + H_c \Sigma yx - V_c \Sigma x^2 + M_c \Sigma x - \Sigma m_L x - H_c \Sigma yx - V_c \Sigma x^2 - M_c \Sigma x = 0 \tag{33}$$

---

* The equation for portion of arch to right of crown is same as equation for portion of arch to left of crown if right portion is assumed as turned end for end, but when returned again to its true position the direction of the deflection will be opposite to what it would be if it were actually the portion of the arch to the left of the crown.

whence if arch is symmetrical

$$V_c = \frac{\Sigma(m_R - m_L)x}{2\Sigma x^2} \tag{34}$$

From Eq. (31),

$$\Sigma m_L y + H_c \Sigma y^2 + V_c \Sigma xy + M_c \Sigma y + \Sigma m_R y + H_c \Sigma y^2 - V_c \Sigma xy + M_c \Sigma y = 0 \tag{35}$$

From Eq. (32),

$$\Sigma m_L + H_c \Sigma y + V_c \Sigma x + \Sigma M_c + \Sigma m_R + H_c \Sigma y - V_c \Sigma x + \Sigma M_c = 0 \tag{36}$$

If the arch is *symmetrical*, and each half arch divided into n parts, ΣM_c for each half $= nM_c$.

We may also, for convenience, let

$$\Sigma m_L + \Sigma m_R = \Sigma m$$

and

$$\Sigma m_L y + \Sigma m_R y = \Sigma my$$

whence, from Eq. (36),

$$M_c = -\frac{\Sigma m + 2H_c \Sigma y}{2n} \tag{37}$$

From Eq. (35),

$$M_c = -\frac{\Sigma my + 2H_c \Sigma y^2}{2\Sigma y} \tag{38}$$

Equating Eqs. (37) and (38) gives

$$\Sigma m \Sigma y + 2H_c(\Sigma y)^2 = n\Sigma my + 2nH_c \Sigma y^2$$

whence

$$H_c = \frac{n\Sigma my - \Sigma m \Sigma y}{2[(\Sigma y)^2 - n\Sigma y^2]} \tag{39}$$

Equations (34), (37), and (39) apply only to a symmetrical arch. In applying them, the various terms apply to one-half of arch except terms involving Σm.

If the arch is unsymmetrical, Eqs. (33), (35), and (36) are applicable, but equations for M_c and H_c will be somewhat more complicated than those derived for the symmetrical arch.

These equations neglect the effect of direct stress upon the arch and consequently give approximate values which have to be corrected by allowing for the stresses due to direct stress. Terms for temperature and shrinkage could readily be included if desired. The author considers that the equations given in Chap. VIII are simpler to apply than the foregoing equations and give more precise results, since they allow for the work due to direct stress.

CHAPTER V

TWO-HINGED PARABOLIC-ARCH RIBS

46. General Condition for Solution.—The horizontal reactions of a two-hinged symmetrical parabolic-arch rib may be readily computed for an arch in which the relation between the depth of the rib and the radius of curvature is such that the effect of curvature upon the distribution of the stress over the cross section may be neglected (see Chap. III), provided either of the following conditions exist.

1. The rib is of constant cross section and so flat that no appreciable error is likely to occur from placing $ds = dx$.
2. The cross section of the rib is so designed that

$$I = I_c \sec \alpha.$$

The following nomenclature is used in this chapter:

Let I = moment of inertia of any cross section of arch.
I_c = moment of inertia of cross section at centre of arch.
M = bending moment at any section of arch due to external forces.
m = bending moment at any section of arch due to horizontal force of unity acting to right at right-hand hinge of arch.
H = horizontal force at either abutment due to abutment thrust or to a tie rod.
Δx_b = horizontal deflection of b (Fig. 1), positive to right.
A = area in square inches of cross section of arch.
A_t = area in square inches of tie rod if one is used to resist horizontal thrust of arch.
E = modulus of elasticity of rib.
E_t = modulus of elasticity of tie rod.
t° = change in temperature in degrees.
ϵ = coefficient of expansion.

47. Determination of Horizontal Component of Reaction Due to Vertical Load.—The horizontal reaction due to any load for the ordinary arch rib is due principally to the tendency of the arch to flatten under the flexure due to the applied load; there is also a secondary horizontal reaction due to the shortening of the arch rib brought about by the axial thrust to which it is subjected.

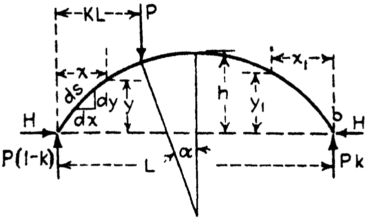


FIG. 1.

The value of the horizontal reaction for a rib such as that shown in Fig. 1 may be determined as follows:

If end b is assumed to be on rollers, its deflection to right, neglecting axial stress, effect of curvature on stress distribution and distortion due to shear is given by the following expression:

$$\Delta x_b = \int \frac{Mmds^*}{EI} \qquad (1)$$

If M_P = bending moment at any section due to applied load P assuming end b on rollers and H removed, the deflection of b to the right due to load P is,

$$\Delta x_b = \int M_P \frac{mds}{EI}$$

But the deflection of b to the right due to the force H alone is

$$\Delta x_b = \int (-Hm)\frac{(m)ds}{EI} = -H\int \frac{m^2ds}{EI}$$

Hence the deflection of b to right resulting from the application of load P and force H is given by the following equation:

$$\Delta x_b = \int M_P \frac{mds}{EI} - H\int \frac{m^2ds}{EI} \qquad (2)$$

Either placing I equal to I_c sec α and noting that $ds = dx$ sec α or placing $ds = dx$ and $I = I_c$ gives the following equations:

$$\Delta x_b = \int \frac{M_P m dx}{EI_c} - H\int \frac{m^2dx}{EI_c} \qquad (3)$$

---

* See Theory of Structures, 3d ed., Art. 177, Spofford.

Substituting values for M_P and m gives

$$\Delta x_b = \frac{1}{EI_c}\int_0^{kL} P(1-k)xy\,dx + \frac{1}{EI_c}\int_0^{L(1-k)} Pkx_1y_1\,dx - \frac{H}{EI_c}\int_0^L y^2\,dx$$

Substituting for y $\frac{4hx}{L^2}(L-x)$ and for y_1 $\frac{4hx_1}{L^2}(L-x_1)$ gives

$$\begin{aligned}\Delta x_b &= \frac{1}{L^2EI_c}\left\{\int_0^{kL} 4P(1-k)hx^2(L-x)\,dx + \int_0^{L(1-k)} 4Pkhx_1^2(L-x_1)\,dx - \frac{H}{L^2}\int_0^L 16h^2x^2(L-x)^2\,dx\right\} = \frac{4Ph(1-k)}{L^2EI_c}\left(\frac{k^3L^4}{3}-\frac{k^4L^4}{4}\right)\\ &\quad + \frac{4Phk}{L^2EI_c}\left[\frac{(L)(L^3)(1-k)^3}{3}-\frac{L^4(1-k)^4}{4}\right] - \frac{H}{EI_c}\left[\frac{16h^2}{L^4}\left(\frac{L^5}{3}-\frac{L^5}{2}+\frac{L^5}{5}\right)\right]\\ &= \frac{4Ph(1-k)(k^3L)}{EI_c}\left(\frac{L}{3}-\frac{kL}{4}\right) + \frac{4PhkL^2}{EI_c}\left[\frac{(1-k)^3}{3}-\frac{(1-k)^4}{4}\right] - \frac{16Hh^2}{EI_c}\left(\frac{L}{3}-\frac{L}{2}+\frac{L}{5}\right) \qquad (4)\\ &= \frac{4Ph(1-k)(kL^2)}{EI_c}\left[\frac{k^2(4-3k)+4(1-k)^2-3(1-k)^3}{12}\right] - \frac{16h^2HL}{30EI_c}\\ &= \frac{Ph}{3}\frac{L^2}{EI_c}(k^4-2k^3+k) - \frac{8}{15}\frac{h^2HL}{EI_c} \qquad (5)\end{aligned}$$

Now, let the axial thrust be assumed to be constant throughout and equal to H,* and assume effect of shrinkage, if a concrete arch, and of temperature to be combined in one term and to equal ϵt°.

Then

$$\Delta x_b = \frac{Ph}{3}\frac{L^2}{EI_c}(k^4-2k^3+k) - \frac{8}{15}\frac{h^2HL}{EI_c} - \frac{HL}{AE} \pm \epsilon t^\circ L \qquad (6)$$

---

* See footnote, Art. 248, Theory of Structures, 3d ed., Spofford; also Art. 53 of this chapter for more accurate method of obtaining effect of axial thrust.

in which the third term equals the horizontal shortening of the rib due to axial thrust.

If this thrust is taken by a rigid abutment, as is usual,

$$\Delta x_b = 0$$

Therefore

$$H = \frac{\frac{Ph}{3}\left(\frac{L}{I_c}\right)(k^4 - 2k^3 + k) \pm \epsilon t^\circ E}{\frac{1}{A} + \frac{8}{15}\frac{h^2}{I_c}} \tag{7}$$

If rib shortening due to axial thrust is neglected, first term in denominator should also be neglected.

If a tie rod is used instead of an abutment to restrain the horizontal movement of the arch as in Fig. 2, the deflection of right end of arch is given by the following expression, in which there is no term for temperature change provided the coefficient of expansion is identical for tie rod and arch rib; and shrinkage of the arch rib, if concrete, is neglected:

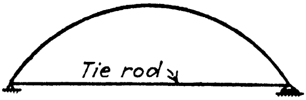


FIG. 2.

$$\frac{Ph}{3}\frac{L^2}{EI_c}(k^4 - 2k^3 + k) - \frac{8h^2HL}{15EI_c} - \frac{HL}{AE} = \frac{HL}{A_tE_t} \tag{8}$$

Equation (8), when solved, gives following value for stress H_t in the rod:

$$H_t = H = \frac{\frac{Ph}{3E}\left(\frac{L}{I_c}\right)(k^4 - 2k^3 + k)}{\frac{1}{AE} + \frac{1}{A_tE_t} + \frac{8h^2}{15EI_c}} \tag{9}$$

The maximum value of all equations for H evidently occurs when $k = \frac{1}{2}$.

From Eqs. (7) and (9), the relation between the values of H for a structure with and without tie rods is as follows if the effect of temperature change is omitted:

$$\frac{H_t}{H} = \frac{\frac{1}{AE} + \frac{8}{15}\frac{h^2}{EI_c}}{\frac{1}{AE} + \frac{1}{A_tE_t} + \frac{8h^2}{15EI_c}} \tag{10}$$

The foregoing ratio is not affected by the loading but depends entirely upon the properties of the arch and tie rod. If the arch rib is of same material as the tie rod, E and E_t may be canceled. All terms should be in same units.

If a uniform load w is used, substitute $wdx = wL(dk)$ for P in Eq. (8), and integrate between appropriate limits. This gives following expression for arch with tie rod, if the latter has the same coefficient of expansion as the rib itself:

$$\int \frac{hw(dk)LL^2}{3EI_c}(k^4 - 2k^3 + k) - \frac{8h^2}{15}\frac{HL}{EI_c} - \frac{HL}{AE} = \frac{HL}{A_tE_t} \quad (11)$$

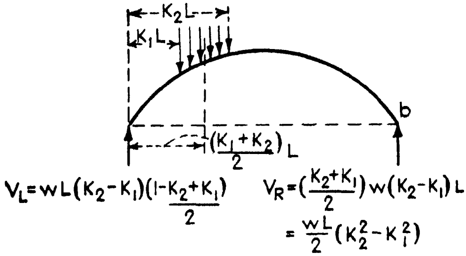


FIG. 3.

Eq. 11, when integrated between limits k_2 and k_1 for a load extending from k_2L to k_1L, as in Fig. 3, gives following value:

$$\frac{hwL^3}{3EI_c}\left(\frac{k_2^5 - k_1^5}{5} - \frac{k_2^4 - k_1^4}{2} + \frac{k_2^2 - k_1^2}{2}\right) - \frac{8}{15}\frac{h^2HL}{EI_c} - \frac{HL}{AE} = \frac{HL}{A_tE_t} \quad (12)$$

For uniform loading extending from left end through distance k_2L, *i.e.*, where $k_1 = 0$,

$$\frac{hwL^3}{30EI_c}(2k_2^5 - 5k_2^4 + 5k_2^2) - \frac{8}{15}\frac{h^2HL}{EI_c} - \frac{HL}{AE} = \frac{HL}{A_tE_t} \quad (13)$$

For uniform loading over entire span, $k_1 = 0$ and $k_2 = 1$.

$$\frac{2hwL^3}{30EI_c} - \frac{8}{15}\frac{h^2HL}{EI_c} - \frac{HL}{AE} = \frac{HL}{A_tE_t} \quad (14)$$

Equations (12), (13), and (14) give following values for H: Load from k_1L to k_2L:

$$H = \frac{\frac{hwL^2}{3EI_c}\left(\frac{k_2^5 - k_1^5}{5} - \frac{k_2^4 - k_1^4}{2} + \frac{k_2^2 - k_1^2}{2}\right)}{\frac{1}{A_tE_t} + \frac{1}{AE} + \frac{8h^2}{15EI_c}} \qquad (15)$$

For load extending from left end through distance k_2L,

$$H = \frac{\frac{1}{30}\frac{hwL^2}{EI_c}(2k_2^5 - 5k_2^4 + 5k_2^2)}{\frac{1}{A_tE_t} + \frac{1}{AE} + \frac{8h^2}{15EI_c}} \qquad (16)$$

For uniform load over entire structure,

$$H = \frac{\frac{1}{15}\frac{hwL^2}{EI_c}}{\frac{1}{A_tE_t} + \frac{1}{AE} + \frac{8h^2}{15EI_c}} \qquad (17)$$

If a tie rod is not used, terms containing A_t should be omitted from all equations. If a tie rod is not used, and the effect of direct stress is neglected, terms containing A and A_t should both be omitted, and Eq. (17) becomes

$$H = \frac{1}{8}\frac{wL^2}{h} \qquad (18)$$

The effect of temperature and shrinkage may be considered as suggested in case A of Art. 44 (Chap. IV).

48. Unsymmetrical Parabolic-arch Rib.—If the arch rib is unsymmetrical, the equations just deduced are not applicable, since the reaction components are dependent upon the relative level of the end pins as well as upon the applied load. If the unsymmetrical condition of the arch rib is due to its being placed on a gradient as in Fig. 4, the rib itself being symmetrical about an axis through the crown normal to line connecting end pins, the arch may be solved for any load by dividing the load into components normal and parallel, respectively, to the line joining the end hinges. The reactions due to the normal components may then be obtained by the use of the equations in Art. 47.

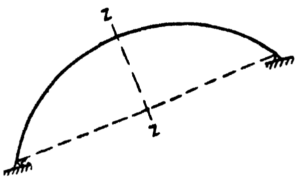


FIG. 4.

To determine the reactions due to the component parallel to the line connecting the end hinges, neglecting the effect of rib shortening and temperature and considering an arch without a tie rod, assume the arch to be rotated to the position shown in Fig. 5.

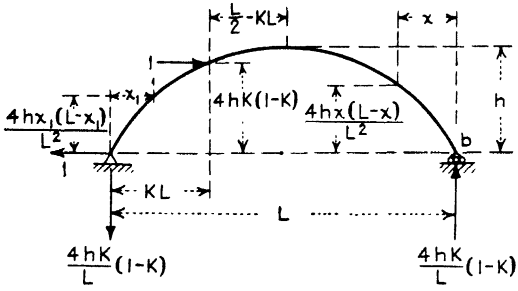


FIG. 5.

Referring to Fig. 5, assuming temporarily that end b is on rollers, and letting Δ_b = deflection of b to right, we get by Eq. (1) for the effect of a load of unity acting as shown:

$$EI_c\Delta_b = \int_0^{L-kL}\left[\frac{4hk(1-k)x}{L}\right]\left[\frac{4hx}{L^2}(L-x)\right]dx$$

$$+ \int_0^{kL}\left[-\frac{4hk(1-k)x_1}{L} + \frac{4hx_1}{L^2}(L-x_1)\right]\left[\frac{4hx_1}{L^2}(L-x_1)\right]dx$$

$$= \frac{16h^2k(1-k)}{L^3}\left\{\int_0^{L-kL} x^2(L-x)dx - \int_0^{kL} x_1{}^2(L-x_1)dx\right\}$$

$$+ \int_0^{kL}\frac{16h^2}{L^4}x_1{}^2(L-x_1)^2dx$$

$$= \frac{16h^2k(1-k)}{L^3}\left\{L\left[\frac{(L-kL)^3-(kL)^3}{3}\right] - \left[\frac{(L-kL)^4-(kL)^4}{4}\right]\right\}$$

$$+ \frac{16h^2}{L^4}\left[\frac{(kL)^3}{3}L^2 - \frac{2L(kL)^4}{4} + \frac{(kL)^5}{5}\right]$$

$$= 16h^2k(1-k)^4L\left[\frac{1}{3} - \frac{(1-k)}{4}\right] -$$

$$16h^2k(1-k)L\left(\frac{k^3}{3} - \frac{k^4}{4}\right) + 16h^2Lk^3\left(\frac{1}{3} - \frac{k}{2} + \frac{k^2}{5}\right)$$

$$= 16h^2k(1-k)^4L\left(\frac{1+3k}{12}\right) - 16h^2k^4(1-k)L\left(\frac{4-3k}{12}\right) + \frac{16h^2Lk^3}{30}(10-15k+6k^2)$$

$$= \frac{8h^2kL}{30}(5-5k-10k^2+20k^3-8k^4) = \frac{8h^2HL}{15}$$

[see Eq. (5)]

Therefore, when point b is fixed in position,

$$H = \tfrac{1}{2}k(5-5k-10k^2+20k^3-8k^4) \qquad (19)$$

49. Position Line. Definition.—The position line for a hinged arch is the line upon which the reactions for any load and the load itself meet. Since in a hinged arch the reactions must pass through the hinges, their magnitude and direction for any load may be readily determined, once the position line is established.

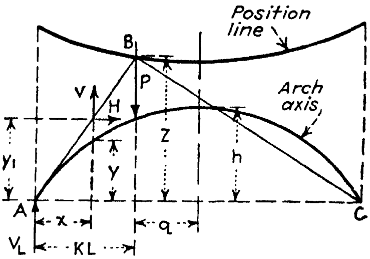


FIG. 6.

50. Position Line. Equation.—The equation for the position line for a symmetrical parabolic arch subjected to a vertical load may be readily determined as follows:

Referring to Fig. 6, the vertical component V_L of the left reaction is given by the following equation:

$$V_L = P(1-k)$$

For arch with tie rod, the value of H is given by Eq. (9). Since the slope of the left reaction equals $z/kL = V_L/H$, it follows that

$$z = \frac{PkL(1-k)\left(\dfrac{1}{AE} + \dfrac{1}{A_tE_t} + \dfrac{8h^2}{15EI_c}\right)}{\dfrac{Ph}{3E}\left(\dfrac{L}{I_c}\right)(k^4-2k^3+k)} = \frac{\dfrac{1}{AE} + \dfrac{1}{A_tE_t} + \dfrac{8h^2}{15EI_c}}{\dfrac{h}{3E}\left(\dfrac{1}{I_c}\right)(1+k-k^2)} \qquad (20)$$

z has its minimum value when $k = \frac{1}{2}$ and its maximum value when k = either 0 or unity.

If the arch has no tie rod, and if the first term in the numerator, which is small, is eliminated, Eq. (20) for the position line becomes

$$z = \frac{8}{5}\frac{h}{1 + k - k^2} \tag{21}$$

In using the position line to give position of loading, as explained later, Eq. (21) may be employed in plotting the position line without serious error if the terms of the numerator involving A and A_t are small in proportion to the third term of the numerator, as is often the case. After a design is determined and the values of A and A_t are known closely, the position line may be revised if it seems desirable.

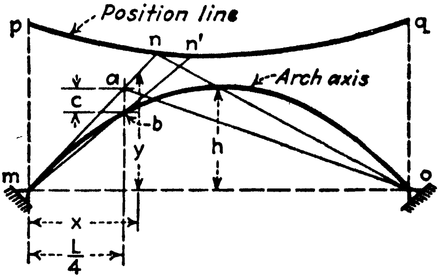


FIG. 7.

51. Determination of Maximum Stress.—Since the maximum fibre stress in an arch rib is due to a combination of thrust and bending moment, the position of the live load that will give maximum fibre stress of either kind at any section of the arch rib is not that which gives maximum bending moment about the neutral axis of the arch rib but instead that which will give maximum moment about an axis passing through the kernel point corresponding to the fibre under consideration.[1]

To determine the position of a uniform load giving maximum moment about an axis passing through a kernel point, proceed as follows:

Let *a* be one of the kernel points at any section of the arch shown in Fig. 7, and *pnq* the position line for vertical loads. Draw *ma* to meet position line at *n*, and draw *no*.

---

[1] See Chap. III.

Since the reactions for any given load intersect on the position line, it follows that *mn* and *no* give the direction of the left and right reactions, respectively, for a vertical load passing through *n*. It is also evident from the diagram that a vertical load passing through *n* causes no moment about the normal to the arch rib passing through kernel point *a*; also that a load to right of *n* causes negative moment about such an axis, while a load to the left of *n* causes positive moment about the axis.

Evidently, for maximum positive moment about *a* the structure should be fully loaded to left of *n*, and for maximum negative moment to right of *n*.

It is of interest to note that, since the moment about a normal axis passing through any point *b* on the *arch axis* due to a uniform load over entire length of span equals zero,[1] the maximum negative moment about such an axis at any section equals maximum positive moment at same section; hence, the maximum negative moment occurs with full load to right of point *n'*, if *b* is a point on the arch axis, and the maximum positive moment occurs with full load to left of same point. For the maximum stress at any given fibre the moment about the kernel point should be used. Note that for sections near the center of span loading for maximum moment should not be continuous from section to end of span. See influence line in Article 55.

To find position of *n* for a specified point, proceed as follows:

Let *a* be the specified point, which is a kernel point of the cross section at point *b*, the quarter point;[2] then $x = L/4$, and

$$y = \tfrac{3}{4}h + c;$$

therefore, equation for *mn* is as follows:

$$\frac{y}{x} = \frac{\tfrac{3}{4}h + c}{L/4}$$

and

$$y = \frac{3hx}{L} + \frac{4cx}{L}$$

Therefore for point of intersection of *mn* and position line,

---

[1] This is true for either a two-hinged or a three-hinged parabolic arch, if the effect of direct stress on H is neglected.

[2] The critical section of a two-hinged parabolic arch without tie rod is at the quarter point or very close thereto, as shown in Art. 54.

$$x\left(\frac{3h}{L} + \frac{4c}{L}\right) = \frac{8}{5}\,\frac{h}{1 + k - k^2}$$

Substitute kL for x, and solve, giving

$$k = \frac{8}{5}\,\frac{h}{1 + k - k^2}\left(\frac{1}{3h + 4c}\right)$$

whence

$$k(1 + k - k^2) = \frac{8}{5}h\left(\frac{1}{3h + 4c}\right)$$

Let $c = \alpha h$. Then

$$k(1 + k - k^2) = \frac{8}{5}\left(\frac{1}{3 + 4\alpha}\right) \tag{22}$$

If $\alpha = 0$,

$$k(1 + k - k^2) = 8/15 \tag{23}$$

which when solved gives

$$k = 0.43 \text{ approximately (see also Art. 54)}$$

Since in all ordinary arches $\alpha = c/h$ is a small fraction, it may be placed equal to 0 without changing the value of k materially.

Equation (23) is correct for the loading giving maximum positive moment about a normal axis passing through the central axis of the arch rib at its quarter point and nearly correct for the loading giving maximum moment about the kernel point *a* which is the loading giving the maximum fibre stress at the lower fibre of the arch rib on a section normal to the arch axis drawn through point *a*.

Substituting preceding value of k for k_2 in Eq. (15), letting $k_1 = 0$, and neglecting terms in denominator due to axial thrust and tie rods, we obtain for maximum $+M$ about a normal axis passing through the central axis at $\frac{1}{4}$ point

$$H = \frac{5wL^2}{8h}\left(\frac{0.43^5}{5} - \frac{0.43^4}{2} + \frac{0.43^2}{2}\right) = 0.049\frac{wL^2}{h}$$

$$V_L = \frac{w(0.43)(1 - 0.215)L^2}{L} = w(0.43)(0.785)L = 0.3375wL$$

$$+M = -0.049\frac{wL^2}{h} \times 0.75h + 0.3375wL \times 0.25L - \frac{w(0.25)^2L^2}{2}$$
$$= +0.0164wL^2$$

For maximum $-M$

$$H = \left(\frac{1}{8} - 0.049\right)\frac{wL^2}{h} = 0.076\frac{wL^2}{h}$$

$$V_L = (0.500 - 0.3375)wL = 0.1625wL$$

$$-M = -0.076\frac{wL^2}{h} \times 0.75h + 0.1625wL^2 \times 0.25 = (-0.057 + 0.0406)wL^2$$

$= -0.0164wL^2$, which verifies the conclusion previously reached that maximum negative moment equals maximum positive moment.

52. Average Axial Thrust. Uniform Loading.—In Eqs. (7) to (17) the value of the axial thrust has been considered to be constant. A somewhat more precise method of obtaining H is to omit the term in the foregoing equations which represents the axial thrust, and to correct the value of H thus determined by the value of H due to axial thrust only. The method which follows shows how this may be done for a uniform load.

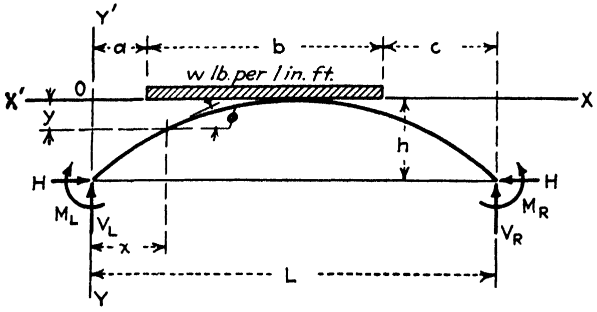


FIG. 8.

The equations developed here hold for any parabolic arch of constant cross section, whether three hinged, two hinged, or no hinged.[1]

Let the X and Y axes intersect at the origin O (Fig. 8), x be positive to the right, and y positive downward. Then the arch axis equation is

$$y = \frac{4h}{L^2}\left(\frac{L}{2} - x\right)^2 \tag{24}$$

---

[1] This method was originally developed by Jon Gunnarsson, S.M., M.I.T., 1930, and extended by A. T. Waidelich, S.M., M.I.T., 1930, to cover cases of uniform loading in any position.

Note that dy/dx is negative for the left half of the arch; hence ϕ will be negative for the left half of the arch and positive for the right half.

General Case.—The partial uniform loading shown is perfectly general; it may be entirely on either half of the arch or extend across the center line as shown.

Let S = total axial length of the arch axis.
T = total axial thrust at any point (x, y).
A = cross-sectional area of arch rib.
T_a = average total axial thrust in entire arch.

Then T_a is such that

$$\frac{T_a S}{AE} = \int \frac{T ds}{AE}$$

Hence

$$T_a = \frac{1}{S} \int T ds$$

Now, noting that $\sin \phi$ for the left half of the arch is negative, the thrust in the various portions of the arch will be for $x \leqq a$:

$$T_1 = H \cos \phi - V_L \sin \phi$$

for $x > a$ and $\leqq (a + b)$:

$$T_2 = H \cos \phi - [V_L - w(x - a)] \sin \phi$$

$(a + b) \leqq x$:

$$T_3 = H \cos \phi - (V_L - bw) \sin \phi$$

But $\sin \phi = \frac{dy}{ds}$; $\cos \phi = \frac{dx}{ds}$;

Therefore

$$\begin{aligned} T_1 ds &= H dx - V_L dy \\ T_2 ds &= H dx - [V_L - w(x - a)] dy \\ T_3 ds &= H dx - (V_L - bw) dy \end{aligned}$$

From Eq. (24):

$$dy = -\frac{8h}{L^2}\left(\frac{L}{2} - x\right) dx$$

Then

$$T_1ds = Hdx - V_Ldy$$

$$T_2ds = Hdx - V_Ldy - \frac{8wh}{L^2}\left(\frac{L}{2} - x\right)(x - a)dx$$

$$T_3ds = Hdx - V_Ldy - \frac{8bwh}{L^2}\left(\frac{L}{2} - x\right)dx$$

Substituting these values in the expression for the average thrust:

$$T_a = \frac{1}{S}\left[\int_{x=0}^{x=a} T_1ds + \int_{x=a}^{x=a+b} T_2ds + \int_{x=a+b}^{x=L} T_3ds\right],$$

$$\begin{aligned} ST_a = H\int_0^a dx - V_L\int_h^{y_a} dy + H\int_a^{a+b} dx - V_L\int_{y_a}^{y_{a+b}} dy &- \\ \frac{8wh}{L^2}\int_a^{a+b}\left(\frac{L}{2} - x\right)(x - a)dx& \\ + H\int_{a+b}^{L} dx - V_L\int_{y_{a+b}}^{h} dy - \frac{8bwh}{L^2}\int_{a+b}^{L}\left(\frac{L}{2} - x\right)dx& \end{aligned}$$

Integrating, we get:

$$\begin{aligned} ST_a = H(a + a + b - a + L - a - b) &- \\ V_L(y_a - h + y_{a+b} - y_a + h - y_{a+b})& \\ - \frac{8wh}{L^2}\left\{\left(\frac{L}{4} + \frac{a}{2}\right)[(a + b)^2 - a^2] - \frac{aL}{2}(a + b - a) -\right.& \\ \left.\frac{1}{3}[(a + b)^3 - a^3]\right\}& \\ - \frac{8bwh}{L^2}\left\{\frac{L}{2}[L - (a + b)] - \frac{1}{2}[L^2 - (a + b)^2]\right\}& \end{aligned}$$

$$\begin{aligned} = HL - \frac{8wh}{L^2}\left[\left(\frac{L}{4} + \frac{a}{2}\right)(a^2 + 2ab + b^2 - a^2) -\right.& \\ \left.\frac{abL}{2} - \frac{1}{3}(a^3 + 3a^2b + 3ab^2 + b^3 - a^3)\right]& \\ - \frac{8bwh}{L^2}\left(\frac{L^2}{2} - \frac{aL}{2} - \frac{bL}{2} - \frac{L^2}{2} + \frac{a^2}{2} + ab + \frac{b^2}{2}\right)& \end{aligned}$$

$$\begin{aligned} = HL - \frac{8wh}{L^2}\left(\frac{abL}{2} + \frac{b^2L}{4} + a^2b + \frac{ab^2}{2} - \frac{abL}{2} -\right.& \\ \left.a^2b - ab^2 - \frac{b^3}{3}\right)& \\ - \frac{8bwh}{L^2}\left(\frac{a^2}{2} + ab + \frac{b^2}{2} - \frac{aL}{2} - \frac{bL}{2}\right)& \end{aligned}$$

Multiplying out and collecting:

$$ST_a = HL + \frac{bwh}{L^2}\left[2bL - \frac{4}{3}b^2 + 4a(L - a - b)\right]$$

$$T_a = \frac{HL}{S} + \frac{bwh}{SL^2}\left(2bL - \frac{4}{3}b^2 + 4ac\right) \tag{25}$$

The average *unit* stress is T_a divided by the area.

Special Cases.—For certain important special cases of uniform loading, the foregoing expression simplifies slightly.

Load extending to either end:

Either $a = 0$ or $c = 0$, and the equation becomes:

$$T_a = \frac{HL}{S} + \frac{2b^2wh}{SL^2}\left(L - \frac{2}{3}b\right) = f_aA \tag{26}$$

Load from one end to center of span:

$b = L/2$ and Eq. (26) becomes

$$\begin{aligned} T_a &= \frac{HL}{S} + \frac{2wh}{SL^2} \cdot \frac{L^2}{4}\left(L - \frac{L}{3}\right) \\ &= \frac{HL}{S} + \frac{whL}{3S} \\ &= \frac{L}{S}\left(H + \frac{wh}{3}\right) = f_aA \end{aligned} \tag{27}$$

Full uniform load:

a and $c = 0$; hence $b = L$, and Eq. (25) gives:

$$T_a = \frac{L}{S}\left(H + \frac{2wh}{3}\right) = f_aA \tag{28}$$

For the case of a full uniform load, if the effect of axial thrust upon H is neglected, we have from Eq. (18) for an arch with rigid abutments

$$H = \frac{1}{8}\frac{wL^2}{h}$$

Therefore

$$wL = \frac{8Hh}{L}$$

which, when substituted in Eq. (28), gives

$$f_a = \frac{1}{AS}HL + \frac{2h}{3}\left(\frac{8Hh}{L}\right)$$

$$= \frac{HL}{AS}\left(1 + \frac{16h^2}{3L^2}\right) = \frac{1}{8}\frac{wL^3}{ASh}\left(1 + \frac{16h^2}{3L^2}\right) \qquad (29)$$

where f_a = average unit stress, neglecting rib shortening. The value of S in preceding equation is given very closely by the equation $S = L\left(1 + \frac{8}{3}\frac{h^2}{L^2}\right)$.[1]

This value of f_a equals the corresponding value for a suspension bridge cable under uniform vertical load per linear horizontal foot over entire cable.

Note that the value of H used in developing Eq. (29) is slightly larger than its true value on account of the neglect of the influence upon the horizontal reaction component of the rib shortening due to axial thrust.

The value of H due to rib shortening resulting from axial thrust may be determined as follows:

Let f_a = average unit stress in rib due to axial thrust. Then, referring to Fig. 1 and assuming arch on rollers at b,

$$\Delta x_b = \frac{f_a L}{E} = \text{horizontal deflection of } b$$

But Δx_b due to horizontal thrust H at b is given by the last term in Eq. (5):

Therefore

$$\Delta x_b = \frac{8}{15}\frac{h^2HL}{EI_c} = \frac{f_a L}{E}$$

---

[1] Let ds = infinitesimal length of arch axis. Then

$$ds = dx\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2}$$

$$= dx\left[1 + \frac{64hx^2}{L^4}\right]^{1/2}$$

Expanding the term in brackets by the binomial theorem, integrating, and neglecting as insignificant all terms except the first two gives

$$S = L\left[1 + \frac{8}{3}\frac{h^2}{L^2}\right]$$

whence

$$H = \frac{15}{8}\frac{f_a I_c}{h^2} \tag{30}$$

Hence H due to rib shortening consistent with f_a as given by Eqs. (26) to (29) may be obtained by substituting in Eq. (30) the value of f_a in foregoing expressions; *e.g.*, H due to rib shortening for uniform load over entire structure is given by following expression:

$$H = \left(\frac{15}{8}\right)\left(\frac{wL^3}{8ASh} \cdot \frac{I_c}{h^2}\right)\left(1 + \frac{16h^2}{3L^2}\right) \tag{31}$$

Problem: Determine the value of H due to rib shortening in a two-hinged symmetrical parabolic-arch rib uniformly loaded throughout its entire length. Arch rib of reinforced concrete: depth, 30 in.; width, 20 in.

w = load per inch. $S = 1.026L$.
L = 100 ft. h = 10 ft.

Neglect reinforcement in computing I and A.

Solution: From Eq. (31)

$$H = \frac{15}{8} w \left(\frac{100}{10}\right)^3\left(\frac{30^2}{12}\right)\left(\frac{316}{8 \times 102.6 \times 12 \times 300}\right) = 15.0w$$

H due to flexure for this case, neglecting effect of axial thrust,

$$= \frac{1}{8} \cdot w\frac{100 \times 12 \times 100 \times 12}{10 \times 12}$$

$$= \frac{1}{8} \cdot 12{,}000w = 1500w$$

hence H due to rib shortening = 15.0/1500 × the value of H with rib shortening neglected; *i.e.*, it reduces H by approximately 1 per cent and reduces the value of f_a by approximately the same amount. This difference is so slight that the correct value of H may be considered as

$$1500w - 15w = 1485w$$

Had conditions been such as to cause the proportional value of H due to rib shortening to be considerably larger than that just determined, a second determination of its value might have been desirable; *e.g.*, in a case where H due to rib shortening was found to be 6⅛ per cent of H with rib shortening neglected, a revised computation using for the total H, 94 per cent of the H neglecting rib shortening gave an average unit stress which, in turn, gave an H due to rib shortening only of almost exactly 6 per cent of that neglecting rib shortening, thus showing the correctness of the assumptions.

53. Thrust at a Specific Section Due to Uniform Load over Entire Span.—To determine total thrust at any specific section due to a uniform load w over entire arch, proceed as follows:

Referring to Fig. 9, let T = normal thrust at a cross section at a lying at right angles to arch axis.

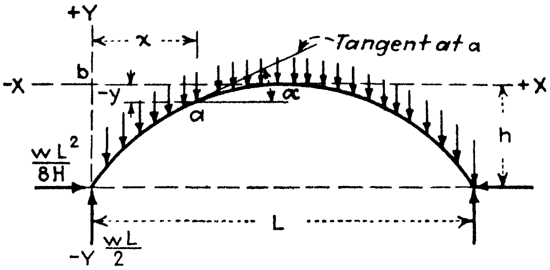


FIG. 9.

The following expression may now be written for T:

$$\frac{T}{\cos\alpha} = \frac{wL^2}{8h} + w\left(\frac{L}{2} - x\right)\tan\alpha \qquad (32)$$

The equation of the parabola referred to point b as an origin with y = ordinate and x = abscissa is

$$y = -\frac{4h}{L^2}\left(\frac{L}{2} - x\right)^2$$

Therefore at point a,

$$\frac{dy}{dx} = \tan\alpha = \frac{8h\left(\frac{L}{2} - x\right)}{L^2} = \frac{4h}{L^2}(L - 2x)$$

Substituting preceding value in Eq. (32) gives

$$\begin{aligned}\frac{T}{\cos\alpha} &= w\left(\frac{L}{2} - x\right)\left(\frac{4h}{L^2}\right)(L - 2x) + \frac{wL^2}{8h} \\ &= \frac{2wh(L - 2x)^2}{L^2} + \frac{wL^2}{8h} \qquad (33)\end{aligned}$$

It is shown in Art. 54 that the maximum moment due to a uniform live load occurs at or near the quarter point; hence it becomes desirable to determine the thrust at this point for a uniform dead load over entire span. This may be done by substituting the value $x = L/4$ in Eq. (33), giving the following result for the thrust for a uniform load over entire structure:

$$\frac{T}{\cos \alpha} = \frac{2wh}{L^2}\left(\frac{L}{2}\right)^2 + \frac{wL^2}{8h} \tag{34}$$

Now, tan α at quarter point equals

$$\frac{8h}{L^2} \cdot \frac{L}{4} = \frac{2h}{L}$$

hence

$$\cos \alpha = \frac{L}{\sqrt{L^2 + 4h^2}}$$

which, when substituted in Eq. (34), gives

$$\begin{aligned} T &= \frac{wL}{\sqrt{L + 4h^2}}\left(\frac{h}{2} + \frac{L^2}{8h}\right) \\ &= \frac{wL}{4}\,\frac{\sqrt{L^2 + 4h^2}}{2h} = \frac{wL}{4 \sin \alpha} \end{aligned} \tag{35}$$

equals thrust at quarter point for uniform load extending over entire arch.

For a uniform load extending from left end to point $0.43L$, the reactions were found in Art. 51 to be as follows:

$$H = 0.049\frac{wL^2}{h} \qquad \text{and} \qquad V_L = 0.3375wL$$

The thrust at the quarter point for this load is, therefore,

$$\begin{aligned} T &= \left[0.049\frac{wL^2}{h} \cos \alpha + (0.3375 - 0.25)wL \sin \alpha\right] \\ &= wL\left(0.049\frac{L}{h} \cos \alpha + 0.0875 \sin \alpha\right) \end{aligned}$$

But at quarter point

$$\frac{L}{2h} = \cot \alpha$$

Therefore

$$\frac{L}{h} = 2 \cot \alpha = \frac{2 \cos \alpha}{\sin \alpha}$$

Therefore

$$T = wL\left(0.049 \times \frac{2\cos^2\alpha}{\sin\alpha} + 0.0875\sin\alpha\right)$$

$$= \frac{wL}{\sin\alpha}(0.098\cos^2\alpha + 0.0875\sin^2\alpha)$$

The thrust corresponding to load extending from right end to point $0.43L$ from left end, which loading gives maximum negative moment at the quarter point as shown in Art. 54, is given by the following expression:

$$T = \left(0.076\frac{wL^2}{h}\cos\alpha + 0.1625wL\sin\alpha\right)$$

$$= wL\left(0.076\frac{L}{h}\cos\alpha + 0.1625\sin\alpha\right)$$

$$= \frac{wL}{\sin\alpha}(0.152\cos^2\alpha + 0.1625\sin^2\alpha)$$

If the live moment is taken about the appropriate kernel point as explained in Art. 51, the value of the thrust at the section need not be computed to obtain the maximum fibre stress, but the equations developed in this article are useful in determining dead stresses and in checking live stresses.

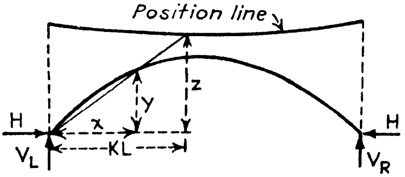


FIG. 10.

54. Location of Section of Maximum Moment.—The location of section of maximum moment due to uniform live load, neglecting axial thrust, may be determined as follows:[1]
From Fig. 10,

$$\frac{x}{y} = \frac{KL}{Z}$$

$$x = \frac{y}{Z}KL$$

---

[1] This development is by I. A. Wojtaszak, S.M., M.I.T., 1930.

$$Z = \frac{8}{5}h\frac{1}{1 + K - K^2}$$

$$y = \frac{4hx}{L}\left(1 - \frac{x}{L}\right)$$

Therefore

$$x = KL\frac{\dfrac{4hx}{L}\left(1 - \dfrac{x}{L}\right)}{\dfrac{8}{5}h\dfrac{1}{1 + K - K^2}}$$

from which

$$x = L\left(1 - \frac{2/5}{K + K^2 - K^3}\right)$$

$$V_L = wLK\left(1 - \frac{K}{2}\right)$$

$$H = \frac{5}{8}\frac{wL^2}{h}\left(\frac{K^2}{2} - \frac{K^4}{2} + \frac{K^5}{5}\right)$$

$$M = V_Lx - Hy - \frac{wx^2}{2}$$

To determine loading for maximum moment differentiate M with respect to K,

$$\frac{dM}{dK} = V_L\frac{dx}{dK} + x\frac{dV_L}{dK} - H\frac{dy}{dK} - y\frac{dH}{dK} - wx\frac{dx}{dK} \tag{36}$$

$$\frac{dx}{dK} = \frac{2}{5}\left[\frac{1 + 2K - 3K^2}{(K + K^2 - K^3)^2}\right]L$$

$$\frac{dV_L}{dK} = wL(1 - K)$$

$$\frac{dy}{dK} = \frac{4h}{L}\left(\frac{dx}{dK} - \frac{2x}{L}\frac{dx}{dK}\right) = \frac{4h}{L}\frac{dx}{dK}\left(1 - \frac{2x}{L}\right) =$$

$$\frac{8h}{5}\frac{1 + 2K - 3K^2}{(K + K^2 - K^3)^2}\left(\frac{4/5}{K + K^2 - K^3} - 1\right)$$

$$\frac{dH}{dK} = \frac{5}{8}\frac{wL^2}{h}(K - 2K^3 + K^4)$$

Substituting the preceding values into Eq. (36),

$$
\begin{aligned}
\frac{dM}{dK} = w\Bigg[L^2K\left(1-\frac{K}{2}\right)\cdot\frac{2}{5}\frac{1+2K-3K^2}{(K+K^2-K^3)^2} &+ \\
L^2(1-K)\left(1-\frac{2/5}{K+K^2-K^3}\right)& \\
-\frac{5}{8}\frac{L^2}{h}\left(\frac{K^2}{2}-\frac{K^4}{2}+\frac{K^5}{5}\right)\frac{8h}{5}\left(\frac{1+2K-3K^2}{(K+K^2-K^3)^2}\right)\left(\frac{4/5}{K+K^2-K^3}-1\right)& \\
-4h\left(1-\frac{2/5}{K+K^2-K^3}\right)\frac{2/5}{K+K^2-K^3}\frac{5}{8}\frac{L^2}{h}(K-2K^3+K^4)& \\
-L^2\left(1-\frac{2/5}{K+K^2-K^3}\right)\frac{2}{5}\frac{1+2K-3K^2}{(K+K^2-K^3)^2}\Bigg]&
\end{aligned}
$$

Let $A = 1 + 2K - 3K^2$.
$B = K + K^2 - K^3$.

$$
\begin{aligned}
0 = \frac{dM}{dK} = wL^2\Bigg[\frac{2}{5}K\left(1-\frac{K}{2}\right)\frac{A}{B^2}+(1-K)\left(1-\frac{2/5}{B}\right) &- \\
\left(\frac{K^2}{2}-\frac{K^4}{2}+\frac{K^5}{5}\right)\frac{A}{B^2}\left(\frac{4/5}{B}-1\right)-\frac{5}{2}\left(1-\frac{2/5}{B}\right)& \\
\left(\frac{2/5}{B}\right)(K-2K^3+K^4)-\left(1-\frac{2/5}{B}\right)\frac{2}{5}\frac{A}{B^2}\Bigg]&
\end{aligned}
$$

Multiplying through by B^3:

$$
\begin{aligned}
0 = wL^2\Bigg[\frac{2}{5}K\left(1-\frac{K}{2}\right)AB+(1-K)B^3-\frac{2}{5}(1-K)B^2& \\
-\frac{4}{5}\left(\frac{K^2}{2}-\frac{K^4}{2}+\frac{K^5}{5}\right)A+\left(\frac{K^2}{2}-\frac{K^4}{2}+\frac{K^5}{5}\right)AB &- \\
(K-2K^3+K^4)B^2+\frac{2}{5}(K-2K^3+K^4)B &- \\
\frac{2}{5}AB+\frac{4}{25}A\Bigg]& \qquad (37)
\end{aligned}
$$

If each term is expanded separately:

$$
\begin{aligned}
A &= 1 + 2K - 3K^2 \\
B &= K + K^2 - K^3 \\
B^2 &= (1 + 2K - K^2 - 2K^3 + K^4)K^2 \\
B^3 &= K^3(1 + 3K - 5K^3 + 3K^5 - K^6) \\
AB &= K(1 + 3K - 2K^2 - 5K^3 + 3K^4)
\end{aligned}
$$

The terms of Eq. (37) become:

$$\frac{2}{5}K\left(1-\frac{K}{2}\right)AB = \frac{K^2}{5}(2+5K-7K^2-8K^3+11K^4-3K^5)$$

$$(1-K)B^3 = K^3(1+2K-3K^2-5K^3+5K^4+3K^5-4K^6+K^7)$$

$$= (K-2K^3+K^4)B^2$$

$$\frac{2}{5}(1-K)B^2 = \frac{2}{5}K^2(1+K-3K^2-K^3+3K^4-K^5)$$

$$= \frac{2}{5}(K-2K^3+K^4)B$$

$$\frac{4}{5}\left(\frac{K^2}{2}-\frac{K^4}{2}+\frac{K^5}{5}\right)A = \frac{2}{25}K^2(5+10K-20K^2-8K^3+19K^4-6K^5)$$

$$\left(\frac{K^2}{2}-\frac{K^4}{2}+\frac{K^5}{5}\right)AB = \frac{K^3}{10}(5+15K-15K^2-38K^3+31K^4+21K^5-25K^6+6K^7)$$

$$\frac{2}{5}AB = \frac{2}{5}K(1+3K-2K^2-5K^3+3K^4)$$

$$\frac{4}{25}A = \frac{4}{25}(1+2K-3K^2)$$

Substituting the foregoing terms in Eq. (37) and collecting terms, and also multiplying through by 50 to eliminate fractions:

$$8-4K-84K^2+75K^3+185K^4-183K^5-156K^6+149K^7+105K^8-125K^9+30K^{10}=0 \qquad (38)$$

Equation (38) gives the value of K for maximum moment due to a uniform live load.

After several trials a value of K may be found which will practically satisfy the preceding equation.

$K = 0.4177000$

$K^2 = 0.1744733$

$K^3 = 0.0728775$

$K^4 = 0.0304409$

$K^5 = 0.0127152$

$K^6 = 0.0053111$

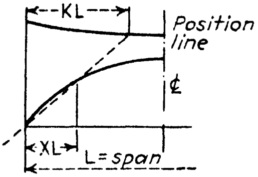
Substituting these values of K in Eq. (38):

$$8-1.670800-14.655756+5.465812+5.631572-2.326877-0.828536+0.330550+0.097298-0.048383+0.004850=-0.000270$$

$K^7 = 0.0022185$
$K^8 = 0.0009266$
$K^9 = 0.0003871$ When $K = 0.4177.$
$K^{10} = 0.0001617$ $X = 0.23L.$

55. Maximum Moments and Influence Tables.—This article contains data that are useful in computing maximum moments on a parabolic-arch rib, and influence lines with the areas included between axis and influence line. The data were computed by advanced students at the Massachusetts Institute of Technology,

Maximum Positive and Negative Moments at Various Sections of a Two-hinged Parabolic Arch for a Uniform Live Load of w Lb. per Ft.



X	*K*	*M*
Section of span	Loaded length	Moment in terms of wL^2 + and −
0.100	0.361	0.0118
0.125	0.371	0.0134
0.150	0.380	0.0148
0.175	0.392	0.0157
0.200	0.403	0.0163
0.225	0.415	0.0165
0.250	0.430	0.0164
0.275	0.443	0.0161
0.300	0.460	0.0153
0.325	0.475	0.0141
0.350	0.492	0.0132
0.375	0.512	0.0120
0.400	0.534	0.0106
0.425	0.378*	0.0079
0.450	0.326*	0.0077
0.475	0.311*	0.0072
0.500	0.306*	0.0073

* These figures refer to loading over intermediate portion of arch. Other figures in this column refer to load from either left end or right end to section of span under consideration.

and the influence lines were prepared by Prof. J. D. Mitsch of the Civil Engineering Department staff of the same institution.

In all cases the effects of direct stress and temperature are omitted, no tie rods are used, and the abutments are considered rigid.

INFLUENCE TABLE FOR MOMENTS ON A TWO-HINGED PARABOLIC ARCH
For moment due to load P multiply tabular values by PL to get moments in foot-pounds, if L is in feet

Points of loading	Moment factors				
	Moments at sections along span				
	$0.10L$	$0.20L$	$0.30L$	$0.40L$	$0.50L$
$0.05L$	+0.034	+0.020	+0.009	−0.000	−0.006
$0.10L$	+0.068	+0.041	+0.019	+0.001	−0.011
$0.15L$	+0.053	+0.062	+0.030	+0.003	−0.015
$0.20L$	+0.038	+0.086	+0.043	+0.008	−0.016
$0.25L$	+0.024	+0.062	+0.059	+0.017	−0.013
$0.30L$	+0.013	+0.038	+0.077	+0.028	−0.009
$0.35L$	+0.002	+0.018	+0.048	+0.042	+0.000
$0.40L$	−0.003	+0.001	+0.024	+0.061	+0.014
$0.45L$	−0.014	−0.013	+0.003	+0.032	+0.032
$0.50L$	−0.020	−0.025	−0.014	+0.013	+0.055
$0.55L$	−0.025	−0.033	−0.027	−0.005	+0.032
$0.60L$	−0.027	−0.039	−0.036	−0.019	+0.014
$0.65L$	−0.028	−0.042	−0.042	−0.024	+0.000
$0.70L$	−0.027	−0.042	−0.043	−0.032	−0.009
$0.75L$	−0.025	−0.038	−0.041	−0.033	−0.013
$0.80L$	−0.022	−0.034	−0.037	−0.031	−0.016
$0.85L$	−0.017	−0.027	−0.030	−0.026	−0.015
$0.90L$	−0.012	−0.019	−0.022	−0.019	−0.011
$0.95L$	−0.006	−0.009	−0.011	−0.010	−0.006

Problems

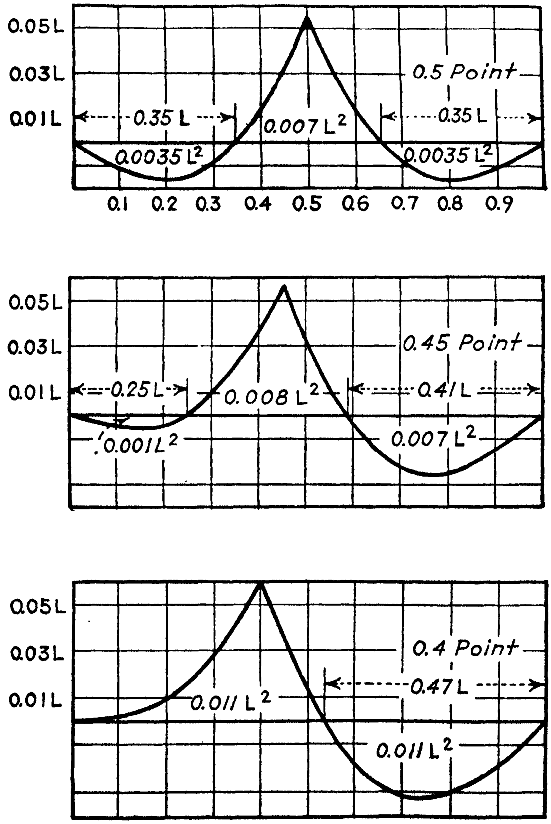
1. Determine the value of the average unit stress in arch of illustrative example in Art. 52 due to uniform load extending from left end to a point $0.43L$ from left end.

2. Compute total thrust at point $0.23L$ from left end of arch of Prob. 1 due to load extending $0.43L$ from left end, and compare with thrust at quarter point due to load extending $0.43L$ from left end.

3. Determine from the influence lines the maximum positive and negative moments at each tenth point along the span for the arch of Prob. 1 due to a uniform live load of 1000 lb. per foot, and verify the results by using the influence table.

4. Using parabolic arch of same cross section as that used in determining location of kernel points in Art. 36 with $E = 29{,}000{,}000$, $L = 100$ ft., and $h = 10$ ft. determine the magnitude and direction of

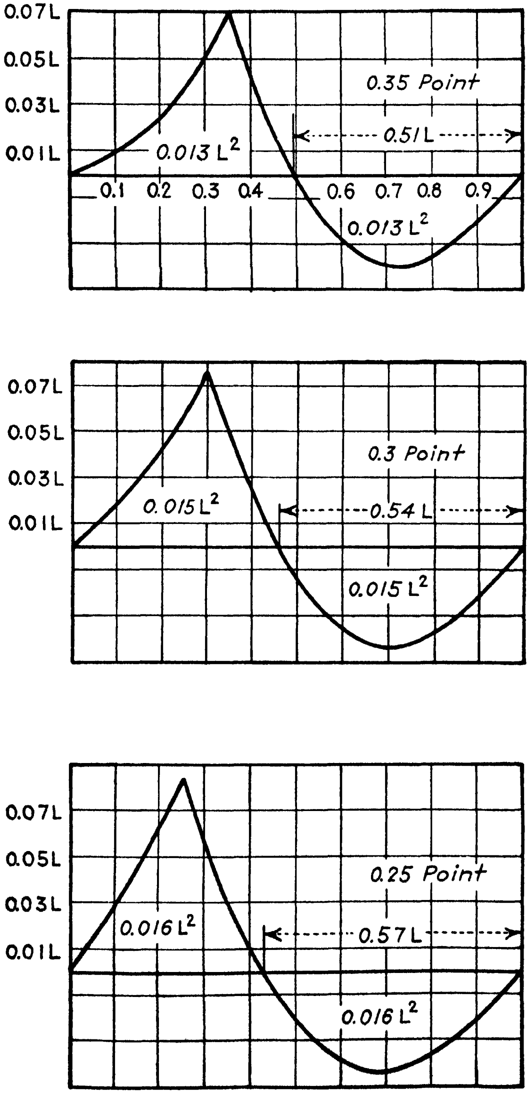
a. The horizontal deflection at left end, assuming this end to be on rollers, due to a vertical force of 50,000 lb. acting at center.



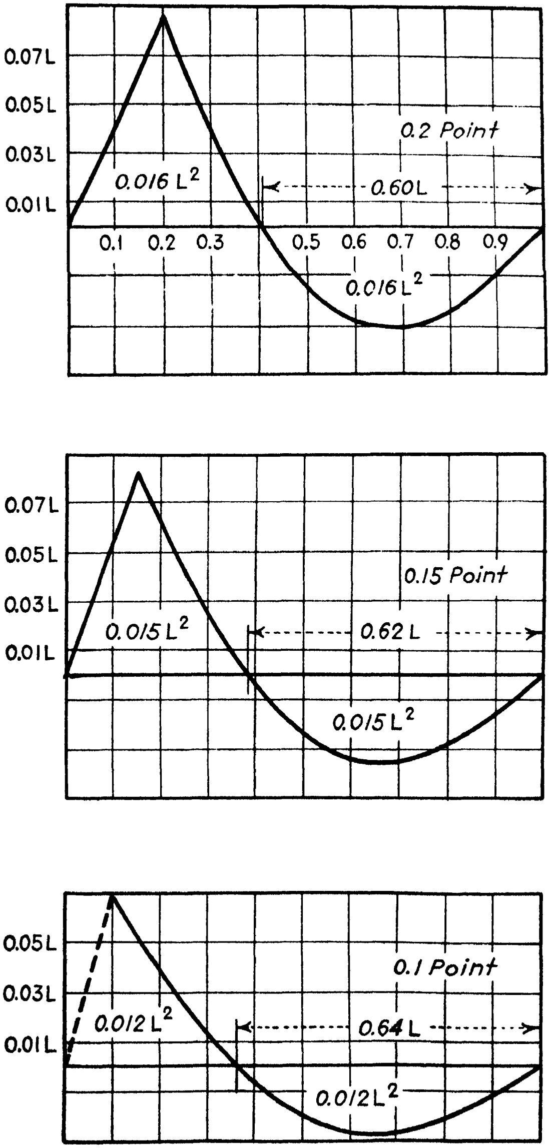
Influence lines for moments for various sections of a two-hinged parabolic arch. Figures between curves and horizontal axes equal areas between curves and axes. Upper figures in right-hand segments designate sections for which curves are drawn.

b. The horizontal component of the reactions due to preceding load, assuming end hinges to be fixed in position.

c. The area of a tie rod required to resist the thrust in the arch due to foregoing load if one end of the arch is on rollers. Assume the tie rod to have a fibre stress of 18,000 lb. (*Problem continued on page* 144.)



Influence lines for moments for various sections of a two-hinged parabolic arch. Figures between curves and horizontal axes equal areas between curves and axes. Upper figures in right-hand segments designate sections for which curves are drawn.



Influence lines for moments for various sections of a two-hinged parabolic arch. Figures between curves and horizontal axes equal areas between curves and axes. Upper figures in right-hand segments designate sections for which curves are drawn.

d. Same as *c* but with a uniform load of 1000 lb. per linear foot applied over the entire span.

e. The maximum fibre stress in arch due to the uniform load in *d*, assuming end hinges to be fixed in position.

f. Assuming structure to be divided into six panels at $16\frac{2}{3}$ ft., each, with the floor at approximately the level of the end hinges, and the intermediate floor beams supported by wire rope hangers connected to the arch rib, compute maximum compressive fibre stress in arch rib due to a uniform live load of W lb. per foot if end hinges are fixed in position.

g. Same as *f*, but use a 20-ton motor truck with 14 tons on rear axle and 6 tons on front axle. Axles are 15 feet center to center, and truck weight is distributed equally to the two arches.

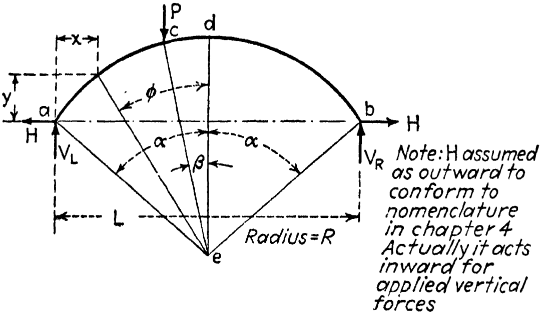
5. Determine effect of rib shortening upon unit stress in arch of Prob. 1 at a section $0.23L$ from left end.

CHAPTER VI

TWO-HINGED SEGMENTAL-ARCH RIBS

56. Reactions.—Equation (1) of Chap. IV may be applied to a segmental two-hinged arch of constant cross section and homogeneous material, as follows:

Omitting for the present the effect of temperature and shrinkage and referring to Fig. 1, let ϕ, α, and β be positive on left of centre and Δx_a = horizontal movement of point a, positive to right.



Note: H assumed as outward to conform to nomenclature in chapter 4 Actually it acts inward for applied vertical forces

FIG. 1.

Substituting values of N_0 and M_0 in the equation, noting that Δx_B and $CD = 0$, and changing signs gives

$$-\Delta x_a = \int_0^L \frac{N}{EA}dx - \int_0^L \frac{M}{EAR}dx + \int_0^S \frac{M}{EI_0}yds - \int_0^S \frac{N}{EAR}yds + \int_0^S \frac{M}{EAR^2}yds$$

which, if E is constant, gives

$$-E\Delta x_a = \left(\frac{1}{I_0} + \frac{1}{AR^2}\right)\int_0^S Myds - \frac{1}{AR}\int_0^L Mdx - \frac{1}{AR}\int_0^S Nyds + \frac{1}{A}\int_0^L Ndx \quad (1)$$

If we let $I_0 = KAR^2$, K being a constant, then $1/AR^2 = K/I_0$

$$\frac{1}{AR} = \frac{KR}{I_0}; \qquad \frac{1}{A} = \frac{KR^2}{I_0}; \qquad \text{and} \qquad \frac{1}{I_0} + \frac{1}{AR^2} = \frac{1}{I_0}(1 + K)$$

therefore, Eq. (1) may be written

$$-I_0E\Delta x_a = (1 + K)\int_0^S Myds - KR\int_0^L Mdx - KR\int_0^S Nyds + KR^2\int_0^L Ndx \quad (2)$$

The values of N and M at any point distant x horizontally from end a are as follows (note that x is positive when to right of a, y positive when above ab, N positive when tension and M positive when causing compression in upper fibre):

Between a and c,

$$N = -V_L \sin \phi + H \cos \phi$$
$$M = V_Lx + Hy = V_LR(\sin \alpha - \sin \phi) + HR(\cos \phi - \cos \alpha)$$

Between b and c,

$$N = V_R \sin \phi + H \cos \phi$$
$$M = V_R(L - x) + Hy = V_RR(\sin \alpha + \sin \phi) + HR(\cos \phi - \cos \alpha)$$

Now,

$$y = R(\cos \phi - \cos \alpha), \qquad ds = -Rd\phi, \qquad dx = -Rd\phi \cos \phi$$

Simplifying Eq. (2) and performing the indicated integrations (see Appendix B) give the following equation:

$$\frac{-2I_0E\Delta x_a}{R^3} = -P[\sin^2 \alpha - \sin^2 \beta + 2 \cos \alpha(\cos \beta - \cos \alpha) - 2(1 + K) \cos \alpha(\alpha \sin \alpha - \beta \sin \beta)] + 2H[3 \sin \alpha \cos \alpha - \alpha - (1 + K)(2\alpha)(\cos^2 \alpha)] \quad (3)$$

Now, let A_0 and C_0 equal the following values which are constants of the structure:

$$A_0 = \sin^2 \alpha - 2 \cos^2 \alpha - 2(1 + K)\alpha \sin \alpha \cos \alpha$$
$$C_0 = \alpha - 3 \sin \alpha \cos \alpha + 2(1 + K)\alpha \cos^2 \alpha$$

Then Eq. (3) becomes

$$\frac{-2I_0E\Delta x_a}{R^3} = -P[(A_0 - \sin^2 \beta + 2 \cos \alpha \cos \beta + 2(1 + K)\beta \cos \alpha \sin \beta)] - 2HC_0 \quad (4)$$

To determine effect of uniform load, w lb. per foot, substitute $wR \cos \beta d\beta$ for P in Eq. (4), and integrate, assuming β to vary between β_n and β_m. The resulting equation will be as follows:

$$\frac{-2I_0E\Delta x_a}{R^3} = -wR\int_{\beta_n}^{\beta_m}[(A_0 \cos \beta d\beta - \sin^2 \beta \cos \beta d\beta$$
$$+ 2 \cos \alpha \cos^2 \beta d\beta + 2(1 + K) \cos \alpha\beta \sin \beta \cos \beta d\beta)] - 2HC_0$$

But

$$\int \cos \beta d\beta = \sin \beta; \qquad \int \sin^2 \beta \cos \beta d\beta = \frac{\sin^3 \beta}{3}$$
$$\int \cos^2 \beta d\beta = \tfrac{1}{2}(\sin \beta \cos \beta + \beta)$$
$$\int \beta \sin \beta \cos \beta d\beta = \int \beta \frac{\sin 2\beta}{2} d\beta = \int \frac{(2\beta)(\sin 2\beta)}{8} d(2\beta)$$
$$= \tfrac{1}{8}(\sin 2\beta - 2\beta \cos 2\beta)$$

Substituting limits

$$\frac{-2I_0E\Delta x_a}{R^3} = -wR[A_0(\sin \beta_m - \sin \beta_n) - \frac{\sin^3 \beta_m - \sin^3 \beta_n}{3}$$
$$+ \cos \alpha(\sin \beta_m \cos \beta_m + \beta_m - \sin \beta_n \cos \beta_n - \beta_n$$
$$+ \frac{2(1 + K)}{8}(\cos \alpha)(\sin 2\beta_m - 2\beta_m \cos 2\beta_m - \sin 2\beta_n$$
$$+ 2\beta_n \cos 2\beta_n)] - 2HC_0 \quad (5)$$

If $\beta_n = 0$, $\sin \beta_n = 0$, and $\cos \beta_n = 1$
Therefore, for this case,

$$\frac{-2I_0E\Delta x_a}{R^3} = -wR\left[A_0 \sin \beta_m - \frac{\sin^3 \beta_m}{3} + \right.$$
$$\cos \alpha(\sin \beta_m \cos \beta_m + \beta_m)$$
$$\left. - \frac{1 + K}{4} \cos \alpha(\sin 2\beta_m - 2\beta_m \cos 2\beta_m)\right] - 2HC_0$$

If $\alpha = 90°$ and $\beta_m = 90°$, which would be the case for a semicircular arch loaded from centre to one end, we obtain

$$\frac{-2I_0E\Delta x_a}{R^3} = -wR\left[A_0 - \frac{1}{3}\right] - 2HC_0$$

But for this case $A_0 = 1$ and $C_0 = \alpha = \pi/2$.

Therefore

$$\frac{-2I_0E\Delta x_a}{R^3} = -\frac{2wR}{3} - H\pi$$

For arch with rigid abutments, $\Delta x_a = 0$
Therefore

$$H = -\frac{2}{3}\frac{wR}{\pi}$$

The foregoing value is for the special case of a semicircular arch on rigid supports, uniformly loaded over half its length. If the uniform load for this case extends over entire span,

$$H = -\frac{4}{3}\frac{wR}{\pi}$$

It follows that the moment M_0 at the centre of a semicircular arch on rigid supports loaded with a uniform load over its entire length is given by the following expression:

$$M_c = \frac{1}{8}w(2R)^2 - \frac{4}{3}\frac{wR^2}{\pi} = 4wR^2\left(\frac{1}{8} - \frac{1}{3\pi}\right)0.070wR^2 = \frac{1}{14}wR^2$$

but

$$R = \frac{L}{2}$$

hence

$$M_c = \tfrac{1}{56}wL^2$$

For a semicircular arch with a concentrated load and rigid supports, Eq. (4) gives the following value for H:

$$H = -\frac{PA_0}{2C_0} = \frac{-P(1 - \sin^2\beta)}{2\pi/2} = -\frac{P\cos^2\beta}{\pi}$$

If P is applied at arch centre, $\beta = 0$.
Therefore for this case, $H = -\dfrac{P}{\pi}$, and

$$M_c = \frac{P}{2}R - \frac{PR}{\pi} = PR\left(\frac{1}{2} - \frac{1}{\pi}\right) = 0.182PR = 0.091PL = \frac{PL}{11}$$

If the ends of the arch rib are connected by a tie rod, one end being on rollers,

$$\Delta x_a = -\frac{H(2R)}{A_t E_t}$$

Substituting this value in Eq. (4) for the general case gives

$$H = -\frac{P[A_0 - \sin^2\beta + 2\cos\alpha\cos\beta + 2(1+K)\beta\cos\alpha\sin\beta]}{\dfrac{4I_0E}{R^2A_tE_t} + 2C_0} \tag{6}$$

If the arch is supported by rigid abutments, the first term in denominator of Eq. (6) = 0, since it corresponds to an infinite value of A_t, and we get

$$H = -\frac{P[A_0 - \sin^2\beta + 2\cos\alpha\cos\beta + 2(1+K)\beta\cos\alpha\sin\beta]}{2C_0} \tag{7}$$

If in Eq. (7) we place H_0 equal to that part of the fraction without the K terms and use C, D, C', G, F, and K to represent constants, we obtain the following equation for an arch without tie rods:

$$-H = \frac{P}{2}\frac{C - C'K}{D + D'K} = \frac{P}{2}\frac{C}{D}\frac{1 - \dfrac{C'}{C}K}{1 + \dfrac{D'}{D}K} = H_0\frac{1 - FK^{*}}{1 + GK} \tag{8}$$

In Eq. (8)

$$F = \frac{C'}{C} =$$

$$\frac{2\cos\alpha(\alpha\sin\alpha - \beta\sin\beta)}{\sin^2\alpha - \sin^2\beta + 2\cos\alpha[(\cos\beta - \cos\alpha) - (\alpha\sin\alpha - \beta\sin\beta)]}$$

$$G = \frac{D'}{D} = \frac{2\alpha\cos^2\alpha}{\alpha - 3\sin\alpha\cos\alpha + 2\alpha\cos^2\alpha}$$

H_0† =

$$\frac{P}{2}\left[\frac{\sin^2\alpha - \sin^2\beta + 2\cos\alpha[(\cos\beta - \cos\alpha) - (\alpha\sin\alpha - \beta\sin\beta)]}{\alpha - 3\sin\alpha\cos\alpha + 2\alpha\cos^2\alpha}\right]$$

---

[*] The effect of placing the K term equal to zero would probably be negligible in any ordinary steel arch rib (see example in Appendix C).

[†] See ANDREWS' "Theory and Design of Structures," for similar equation.

Constants applicable to any arch may be determined for F, G, and H_0 and are given in Table I on the following page.

If $P = 0$, Eq. (4) becomes

$$\frac{I_0 E \Delta x_a}{R^3} = HC_0 \qquad (9)$$

57. Temperature and Shrinkage.—Since the foregoing equation gives the deflection due to a horizontal force H, it may be used to determine the effect of temperature or shrinkage as shown below.

The movement Δx_a of point a due to a change in temperature, assuming one end to be on rollers and abutment to be rigid, is given by the following expression:

Let ϵ = coefficient of linear expansion.

t = change of temperature in degrees, positive when an increase.

L = span.

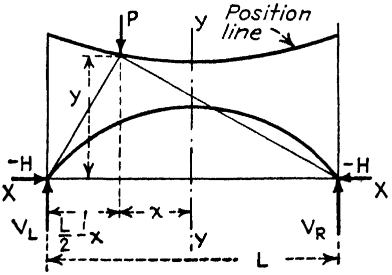


FIG. 2.

Then

$$\Delta x_a = -L\epsilon t = -2R \sin\alpha\epsilon t$$

Therefore

$$2R \sin\alpha\epsilon t = -\frac{R^3 HC_0}{I_0 E}$$

Therefore

$$H = -\frac{2I_0 E \sin\alpha\epsilon t}{R^2 C_0} \qquad (10)$$

If temperature change is an increase, H is negative, indicating thrust. If temperature change is a decrease, H is positive, indi-

TABLE I.—SEGMENTAL ARCH

Tabular values for substitution in $-H = H_0 \dfrac{1 - FK}{1 + GK}$

	β/α	$\alpha = 10°$	$\alpha = 20°$	$\alpha = 30°$	$\alpha = 40°$	$\alpha = 50°$	$\alpha = 60°$	$\alpha = 70°$	$\alpha = 80°$	$\alpha = 90°$	Multiplier*
H_0	0.0	0.3914	0.3880	0.3848	0.3799	0.3733	0.3645	0.3530	0.3375	0.3183	$L/2h$
	0.2	0.3725	0.3690	0.3665	0.3596	0.3518	0.3416	0.3282	0.3108	0.2879	
	0.4	0.3178	0.3137	0.3087	0.3013	0.2912	0.2779	0.2604	0.2397	0.2083	
	0.6	0.2317	0.2275	0.2218	0.2134	0.2019	0.1869	0.1674	0.1142	0.1100	
	0.8	0.1224	0.1191	0.1146	0.1080	0.0992	0.0877	0.0812	0.0542	0.0304	
	1.0	0	0	0	0	0	0	0	0	0	
F	0.0	1.1938	1.1738	1.1383	1.0831	1.0012	0.8811	0.7029	0.4315	0	$L^2/4h^2$
	0.2	1.2038	1.1840	1.1450	1.0949	1.0141	0.8950	0.7170	0.4430	0	
	0.4	1.2339	1.2156	1.1828	1.1314	1.0541	0.9385	0.7622	0.4774	0	
	0.6	1.2881	1.2718	1.2426	1.1963	1.1260	1.0188	0.8486	0.5604	0	
	0.8	1.3702	1.3587	1.3347	1.2967	1.2385	1.1482	1.0004	0.7255	0	
	1.0	1.5041	1.4874	1.4706	1.4461	1.4068	1.3527	1.2755	1.1640	0	
G	...	0.4636	0.4413	0.4065	0.3570	0.2925	0.2141	0.1265	0.0432	0	$L^4/16h^4$

* To obtain actual values of H_0, F, G, multiply tabulated values by $L/2h$, $L^2/4h^2$, $L^4/16h^4$, respectively in which h = rise of arch.

It should be observed that the value of $\dfrac{1 - FK}{1 + GK}$ is approximately unity in numerous cases. It is probable that this fraction may be omitted in most cases.

cating a pull. Shrinkage may be combined with temperature changes as previously stated.

If tie rods are used, no temperature stress will be developed provided arch rib and tie rods have same coefficient of expansion, which is the case if both are of the same material and is nearly correct if rib is of concrete, and tie rod of steel.

58. Position Line.—An expression for the ordinate to the position line in terms of the constants previously deduced may be readily obtained as follows:

Let y and x (Fig. 2) represent coordinates of any point on position line referred to two axes XX passing through the two hinges and YY passing through centre of the arch and normal to XX. Now

$$y = \left(\frac{L}{2} - x\right)\frac{V_L}{-H} = \left(\frac{L}{2} + x\right)\frac{V_R}{-H}$$

also

$$V_L = \frac{\left(\frac{L}{2} + x\right)}{L}P \qquad \text{and} \qquad V_R = \frac{\frac{L}{2} - x}{L}P$$

Therefore

$$y = \left[\left(\frac{L}{2}\right)^2 - (x)^2\right]\frac{P}{-HL}$$

also

$$\frac{L}{2} = R \sin \alpha \qquad \text{and} \qquad x = R \sin \beta$$

Therefore

$$\left(\frac{L}{2}\right)^2 - x^2 = R^2(\sin^2 \alpha - \sin^2 \beta)$$

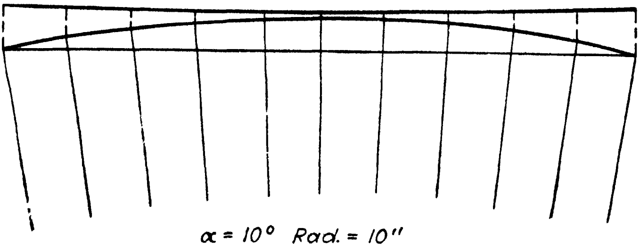
Therefore

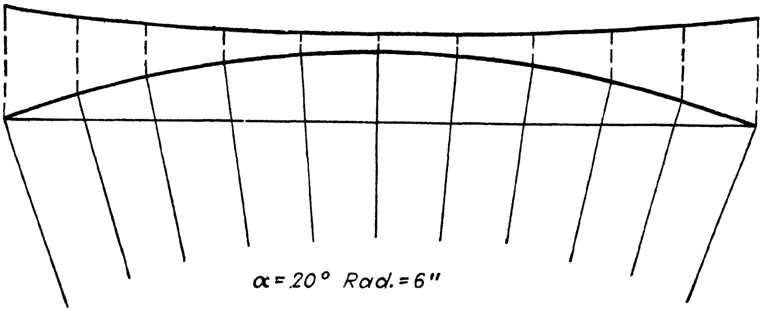
$$y = \frac{\sin^2 \alpha - \sin^2 \beta}{2 \sin \alpha}\left(\frac{PR}{H_0}\right)\frac{1 + GK}{1 - FK} = y_0\frac{1 + GK}{1 - FK}$$

A table of constants could be readily prepared for y by the use of the constants in the table previously prepared.

If $\frac{1 + GK}{1 - FK}$ is taken as unity,

$$y = y_0 = \frac{\sin^2 \alpha - \sin^2 \beta}{2 \sin \alpha}\left(\frac{PR}{H_0}\right)$$





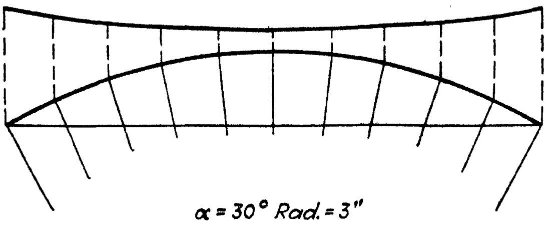


FIG. 3.—Position lines for two-hinged segmental arches.

Table II on page 154 contains values of y_0/PR for various values of α/β and Figs. 3, 4, and 5 show position lines to full size for various cases. These lines were constructed by J. D. Mitsch, assistant

TABLE II.—ORDINATES TO POSITION LINES FOR TWO-HINGED SEGMENTAL ARCHES FOR CASE WHERE $(1 + GK)/(1 - FK)$ EQUALS UNITY

	Values of y_0/PR					
β/α	$\alpha = 10°$	$\alpha = 20°$	$\alpha = 30°$	$\alpha = 40°$	$\alpha = 50°$	$\alpha = 60°$
0	0.0195	0.0770	0.1741	0.3078	0.4784	0.6861
0.2	0.0196	0.0777	0.1755	0.3101	0.4815	0.6896
0.4	0.0201	0.0802	0.1793	0.3168	0.4901	0.7020
0.6	0.0210	0.0835	0.1867	0.3284	0.5085	0.7223
0.8	0.0223	0.0867	0.1977	0.3448	0.5328	0.7516
1.0	0.0234	0.0965	0.2140	0.3731	0.5682	0.7922

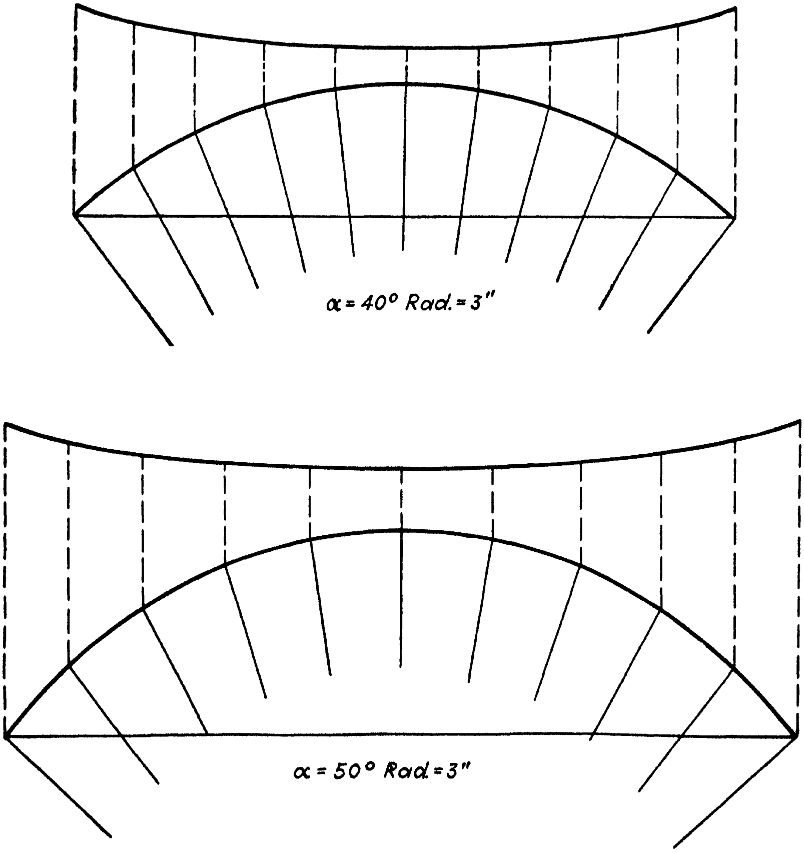


FIG. 4.—Position lines for two-hinged segmental arches.

professor of civil engineering, Massachusetts Institute of Technology.

For case where $\alpha = 90°$, $H_0 = H$ since F and G each equal 0, and therefore

$$H_0 = H = -\frac{P\cos^2\beta}{\pi}$$

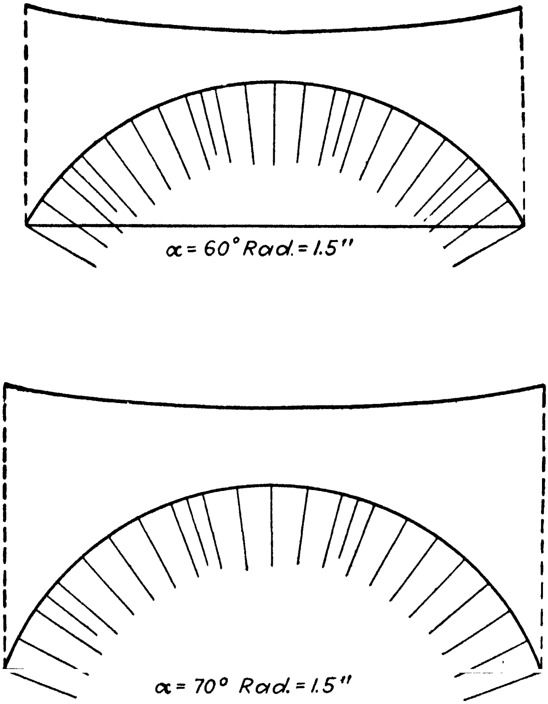


FIG. 5.—Position lines for two-hinged segmental arches.

therefore
$$y = y_0 = \frac{\cos^2\beta}{2}\,\frac{PR\pi}{P\cos^2\beta} = \frac{\pi R}{2}$$

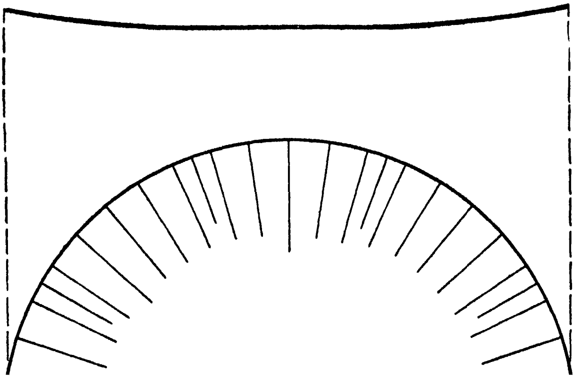
hence the position line is a straight line parallel to the axis XX and distant $\pi R/2$ from it.

For case where $\beta = \alpha$, $H_0 = 0$.

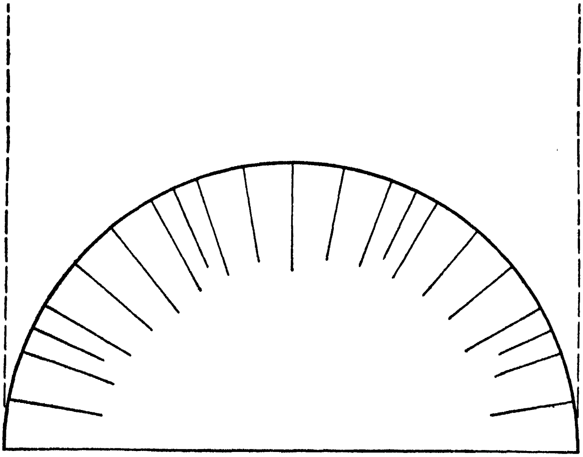
Therefore expression for $Y_0 = 0/0$ and is indeterminate.

For solution of this indeterminate case, see Appendix B.

The position line may be used to determine the loading for maximum functions of the external forces as in the parabolic arch.



$\alpha = 80^\circ$ Rad. = 1.5″



$\alpha = 90^\circ$ Rad. = 1.5″

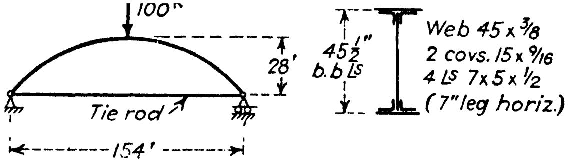
Fig. 6.—Position lines for two-hinged segmental arches.

Problems

1. Determine H for a two-hinged segmental-arch rib due to a vertical load of 100,000 lb. at centre. $\alpha = 30^\circ$, $R = 100$ ft. Cross section of arch same as in illustrative problem in Art. 52. Divide half arch into 15 sections. Assume $\frac{1 - FK}{1 + GK}$ = unity.

2. Determine H for same arch due to temperature change of +60°F.

3. Same as Prob. 1, except that horizontal reactions are resisted by tie rods.



4. Arch shown in figure is a segmental steel arch with steel tie rod. Assume $I_0 = I$; $K = 0$.

a. Considering arch as a two-hinged arch with ends fixed and no tie rod, compute H for a load of 100 kips at the centre of the span.

b. Assuming the unit stress in the tie rod to be 16,000 lb. per square inch, find the area required in the rod, the only load being the 100 kips at the centre.

Using a tie rod of the area computed above:

c. Find the total stress in the tie rod for a unit load at an assigned point. Divide arch into 44 sections of 3.5′ each.

d. By integration, find the total stress in the rod for a uniform load of 1000 lb. per foot over the entire span.

5. Using a two-hinged segmental arch of six panels of same cross section and dimensions as the parabolic arch in Prob. 4 of Chap. V, construct influence lines for the maximum compression and tension at the crown, and compute the maximum compression and tension due to a uniform load of 1000 lb. per foot.

CHAPTER VII

DEFLECTION OF TWO-HINGED AND THREE-HINGED ARCH RIBS

59. Statement of Conditions.—The determination of the deflection of arch ribs having either two or three hinges may be divided into two stages:

A. Deflection due to flexure of the member, assuming its axial length to remain constant. (Deflection due to shear is small and is neglected.)

B. Deflection due to change in axial length of member caused by applied forces, by change in temperature, or by settlement of the supports. It should be observed that in the case of an arch rib with no intermediate hinge, this deflection is partially due to the bending of the rib resulting from changing the length of a bent bar when its ends are fixed in position.

Of the two cases, case *A* is generally the more important. However, in the case of a symmetrical parabolic two-hinged arch under *uniform loading over its entire length,* no deflection occurs except that noted under case *B*.

60. Deflection Due to Flexure. *Case A*.—To determine the deflection under case *A*, the general formula for beam deflection is applicable if ds is substituted for dx (see also Art 47). The formula thus modified will be

$$\delta = \int \frac{Mmds}{EI} \quad (1)$$[1]

In the foregoing equation:

ds = length of an infinitesimal section of the arch rib measured along the axis.

δ = deflection of the arch rib at any point in any direction.

---

[1] This formula is strictly applicable only when it may be assumed that the distribution of stress over the cross section of the curved bar is similar to that over the cross section of a straight bar, a condition that is essentially true for all ordinary bridge arches.

I = moment of inertia of cross section of arch rib normal to axis at any point.
M = bending moment at any section due to applied forces.
m = numerical coefficient of bending moment at any section due to a load of unity acting at the point and in the direction of the desired deflection.
E = modulus of elasticity of material.
s = length of the portion of the bar under consideration.

Equation (1) is not integrable for arches of ordinary types. It may, however, be made applicable to any arch by expressing it as a summation, thus:

$$\delta = \sum \frac{Mm\Delta s}{EI} \tag{2}$$

For most hinged-bridge arches, certain approximations or predeterminations of shape of the arch rib may be made which will put Eq. (1) into integrable form. Examples of these in common use follow.

61. Arches of Constant Cross Section. *Case A.*—For the case of *flat*-arch ribs of *constant cross section,* substitute dx for ds, getting the following equation:

$$\delta = \int \frac{Mmdx}{EI} \tag{3}$$

It should be noted that Eq. (3) is identical in form with the expression for beam deflection, but M and m would have different values in an arch than in a beam owing to the effect of the abutment thrust.

62. Arches of Variable Cross Section. *Case A.*—Referring to Fig. 1:

Let α = angle with the vertical made by any given section.
I_c = moment of inertia of cross-section of arch rib at crown.

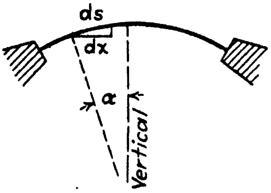


FIG. 1.

Now, let the section of the arch rib be varied so that $I = I_c \sec \alpha$. Then, since $dx = ds \cos \alpha$, ds/I will equal dx/I_c. Therefore

$$\delta = \int \frac{Mmds}{EI} = \int \frac{Mmdx}{EI_c} \tag{4}$$

63. Application of General Equation. *Case A.*—Let the arch rib shown in Fig. 2 be either two hinged or three hinged.

Let δ_{yq} = downward deflection at point q due to load P.

H_p = horizontal component of either reaction due to load P.

H_q = horizontal component of either reaction due to load unity acting downward at q.

x = horizontal distance from point a to a vertical section through any point on the arch axis; positive to right.

y = vertical distance from axis ab to same point on arch axis; positive when upward.

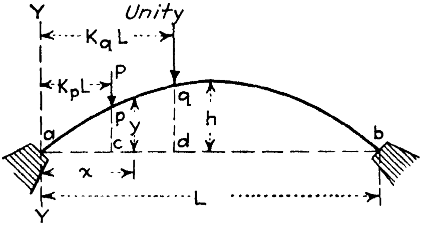


FIG. 2.

Let the load P be applied at distance k_pL from left support, and the load of unity at distance k_qL from left support, k_q being greater than k_p.

Let H_p and H_q be positive when acting toward centre of span.

We may now obtain from Eq. (4) the following general expression for δ_{yq} when the load is between point q and the left support. The complete derivation of this formula is contained in Appendix D.

$$\delta_{yq} = \frac{Pk_pL^3}{6EI_c}(k_q^3 - 3k_q^2 + 2k_q + k_qk_p^2 - k_p^2)$$

$$+ \frac{PH_q}{EI_c}\left[k_pL\int_0^{k_pL} ydx - \int_0^{k_pL} xydx - k_pL\int_0^L ydx + k_p\int_0^L xydx\right]$$

$$+ \frac{H_p}{EI_c}\left[k_qL\int_0^{k_qL} ydx - \int_0^{k_qL} xydx - k_qL\int_0^L ydx + k_q\int_0^L xydx\right]$$

$$+ \frac{H_pH_q}{EI_c}\int_0^L y^2dx \quad (5)$$

It should be observed that the first term of foregoing equation gives the deflection of any point of an end-supported beam of constant section due to a concentrated load located at any point, the remaining terms giving the deflection in the curved beam due to the horizontal thrust, the value of the latter being determined by methods previously given. It should also be observed that the deflection when point q is between the load and the left end may be obtained by transposing the suffixes q and p.

Equation (5) is applicable to either a two- or a three-hinged arch.

The value of the integrals of Eq. (5) may be determined by integration in some cases; for arches of such shape that the terms are difficult or impossible to integrate, the methods given below may be applied.

$\int_0^L ydx$ = area $aqba$ which may be determined by a planimeter or by summation of areas.

$\int_0^L xydx$ = statical moment of area $aqba$ about axis yy. If the arch is symmetrical, the area $aqba$ may be multiplied by $L/2$ to obtain the value of this integral, the area being determined by computation or by planimeter; otherwise, the integral may be determined by dividing the area into vertical strips and summing up the statical moment of each strip about yy.

$\int_0^L y^2dx$ = twice the statical moment of area $aqba$ about axis ab. This may be determined directly by an integraph or approximately as above except that the statical moment of the vertical strips should be taken about ab.

$\int_0^{k_pL} ydx$ or $\int_0^{k_qL} ydx$ = area apc or area aqd which may be determined by a planimeter or by summation of areas.

$\int_0^{k_pL} xydx$ or $\int_0^{k_qL} xydx$ = statical moment of area apc or aqd, respectively, about axis yy, which may be determined as in the second case except for the difference in the area.

64. Deflections of a Parabolic-arch Rib. *Case A.*—Equation (5) can be integrated without difficulty for the case of a parabolic arch by substituting for y its value, *viz.*, $\frac{4hx}{L^2}(L - x)$. The various steps in the substitutions are given in detail in Appendix D. The final result for the downward deflection at any point q due to a load P applied at any other point p located between q and the left support is, in the case of a two- or three-hinged parabolic arch, as follows:

$$\begin{aligned}\delta_{vq} = \frac{Pk_pL^3}{6EI_c}(k_q^3 - 3k_q^2 + 2k_q + k_qk_p^2 - k_p^2) + \\ \frac{H_phk_qL^2}{3EI_c}(2k_q^2 - k_q^3 - 1)^* \\ + \frac{8}{15}\frac{H_pH_qh^2L}{EI_c} + \frac{H_qhk_pPL^2}{3EI_c}[2k_p^2 - k_p^3 - 1]^* \quad (6)\end{aligned}$$

65. Deflections of a Two-hinged Parabolic-arch Rib, Concentrated Load. *Case A.*—For the case under consideration, the horizontal components of the reactions, neglecting effect of axial thrust, are found from Eq. (7), Chap. V, to be as follows:

$$H_p = \frac{5}{8}\frac{PLk_p}{h}(1 - 2k_p^2 + k_p^3)$$

$$H_q = \frac{5}{8}\frac{Lk_q}{h}(1 - 2k_q^2 + k_q^3)$$

These reaction components are always positive for downward loads and act toward centre of span.

Substituting these values in Eq. (6) gives the following expression:

$$\begin{aligned}\delta_{vq} = \frac{PL^3}{24EI_c}[4k_p(1 - k_q)(2k_q - k_q^2 - k_p^2) \\ - 5k_pk_q(1 - 2k_p^2 + k_p^3)(1 - 2k_q^2 + k_q^3)] \quad (7)\end{aligned}$$

In practice it is seldom necessary to determine the deflection at places other than the centre and the quarter points; hence, it is convenient to have special equations for these points. The following have been obtained by making proper substitutions in Eq. (7):

---

* Each of these terms = 0 when $k = 1$ and otherwise is negative, since in order that such a term may be positive $k^2(2 - k)$ must be greater than 1, which is impossible for any fractional value of k.

Deflection at center:

$$\delta_{yq} = \frac{PL^3}{384EI_c}(-k_p + 18k_p^3 - 25k_p^4) \qquad (8)$$

Deflection at quarter point with load to *left* of point:

$$\delta_{yq} = \frac{PL^3}{16 \times 384EI_c}(51k_p - 198k_p^3 - 285k_p^4) \qquad (9)$$

Deflection at quarter point with load to *right* of point:

$$\begin{aligned}\delta_{yq} &= \frac{PL^3}{16 \times 384EI_c}(243k_p - 768k_p^2 + 826k_p^3 - 285k_p^4 - 16) \\ &= \frac{PL^3}{16 \times 384EI_c}(1 - k_p)(227k_p - 541k_p^2 + 285k_p^3 - 16) \qquad (10)\end{aligned}$$

The influence tables which follow give the deflections at quarter points and centre for load of unity at various points on the span. From these tables influence lines for deflections may be constructed from which the deflection at these points due to flexure only with loads in any position may be obtained by scale.

Influence Table for Deflections Due to Curvature, Two-hinged Flat Parabolic Arch

All tabular values to be multiplied by $PL^3/384EI_c$.
δ_1 = deflection at left quarter point ($k_q = 0.25$)
δ_2 = deflection at center ($k_q = 0.5$)
+ indicates downward deflection.

k_p	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
δ_1	+0.3046	+0.5100	+0.4858	+0.2430	−0.0664	−0.3250	−0.4582	−0.4340	−0.2634
δ_2	−0.0845	−0.0960	−0.0165	+0.1120	+0.1875	+0.1120	−0.0165	−0.0960	−0.0845

66. Deflections of a Three-hinged Parabolic-arch Rib, Concentrated Load. *Case A.*—It will be sufficient to develop expressions for deflection due to load on left half only. For this case

$$H_p = \frac{Pk_pL}{2h}$$

If q is on left half of arch, $H_q = k_qL/2h$.

If q is on right half of arch, $H_q = (1 - k_q)\frac{L}{2h}$.

The deflection of point q when q is on the left half of the arch between the load and the crown, *i.e.*, when k_q is greater than k_p but less than $L/2$, may be found by substituting the value of H_p and the proper value of H_q in Eq. (6). The resulting expression is as follows:

$$\delta_{vq} = \frac{Pk_pL^3}{30EI_c}[5(1 - k_q)(2k_q - k_q^2 - k_p^2) + 5k_q(2k_q^2 - k_q^3 + 2k_p^2 - k_p^3) - 6k_q] \quad (11)$$

The deflection of point q when q is on the left half of the arch and between the load and the left support, *i.e.*, when k_q is less than k_p, may be obtained by transposing the suffixes q and p, in the foregoing equation, points q and p again being both on left half of arch.

The deflection of any point q on the right half of the arch, load P being on the left half, may be found by substituting in Eq. (6) the proper values of H_p and H_q. The resulting equation is as follows:

$$\delta_{vq} = \frac{Pk_pL^3}{30EI_c}[(1 - k_q)(10k_q - 5k_q^2 + 5k_p^2 - 5k_p^3 - 1) + 5k_q(2k_q^2 - k_q^3 - 1)] \quad (12)$$

Proper substitutions in Eqs. (11) and (12) give the following expressions for deflection at the crown and quarter points, respectively, load P being always on *left half* of arch.

Deflection at crown:

$$\delta_{vq} = \frac{PL^3}{480EI_c} \cdot k_p[-3 + 40k_p^2(1 - k_p)] \quad (13)$$

Deflection at quarter point, load to left of point:

$$\delta_{vq} = \frac{PL^3}{480EI_c} \cdot \frac{k_p}{16}[71 - 320k_p^2(1 + k_p)] \quad (14)$$

Deflection at quarter point, load to right of point:

$$\delta_{vq} = \frac{PL^3}{480EI_c} \cdot \frac{1}{16}[-20 + 311k_p - 320k_p^2(3 - 3k_p + k_p^2)] \quad (15)$$

Deflection at three-quarter point (*i.e.*, at quarter point on right half of arch):

$$\delta_{vq} = \frac{PL^3}{480EI_c} \cdot \frac{k_p}{16}[-49 + 320k_p^2(1 - k_p)] \quad (16)$$

The following table is computed from Eqs. (13), (14), (15), and (16). The influence lines in Figs. 3 and 4 are plotted from these tables and give the deflection at center and quarter points from loads in any position on the span.

INFLUENCE TABLE, DEFLECTIONS DUE TO CURVATURE, THREE-HINGED FLAT PARABOLIC ARCH

All tabular values to be multiplied by $PL^3/480EI_c$.

δ_1 = deflection at left quarter point (k_q = 0.25)

δ_2 = deflection at center (k_q = 0.5)

+ indicates downward deflection.

k_p	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
δ_1	+0.4218	+0.6955	+0.6393	+0.2530	−0.2813	−0.4570	−0.5408	−0.4845	−0.2883
δ_2	−0.2640	−0.3440	−0.1440	+0.3360	+1.0000	+0.3360	−0.1440	−0.3440	−0.2640

67. Deflection Due to Uniform Load and to Concentrated Load System. *Case A.*—The deflections due to a uniform load may be obtained from the various equations previously developed by placing $wdx = P$, $x = k_pL$, and integrating between proper limits. The application of this method to Eqs. (8) and (13) is illustrated below and shows that there is no deflection at the crown due to flexure (case A) for either two-hinged or three-hinged parabolic arches when the uniform load extends over the entire span.

Making the foregoing substitution in Eqs. (8) and (13), integrating between limits $L/2$ and 0, and doubling the values give the following results:

From Eq. (8),

$$\delta_{vq} = 2\int_0^{\frac{L}{2}} \frac{wdxL^3}{384EI_c}\left(-\frac{x}{L} + \frac{18x^3}{L^3} - 25\frac{x^4}{L^4}\right) = 0$$

From Eq. (13),

$$\delta_{vq} = 2\int_0^{\frac{L}{2}} \frac{wdxL^3}{480EI_c}\left(-\frac{3x}{L} + 40\frac{x^3}{L^3} - 40\frac{x^4}{L^4}\right) = 0$$

The deflection at quarter and center points due to uniform loading over any other portion of the span may be determined

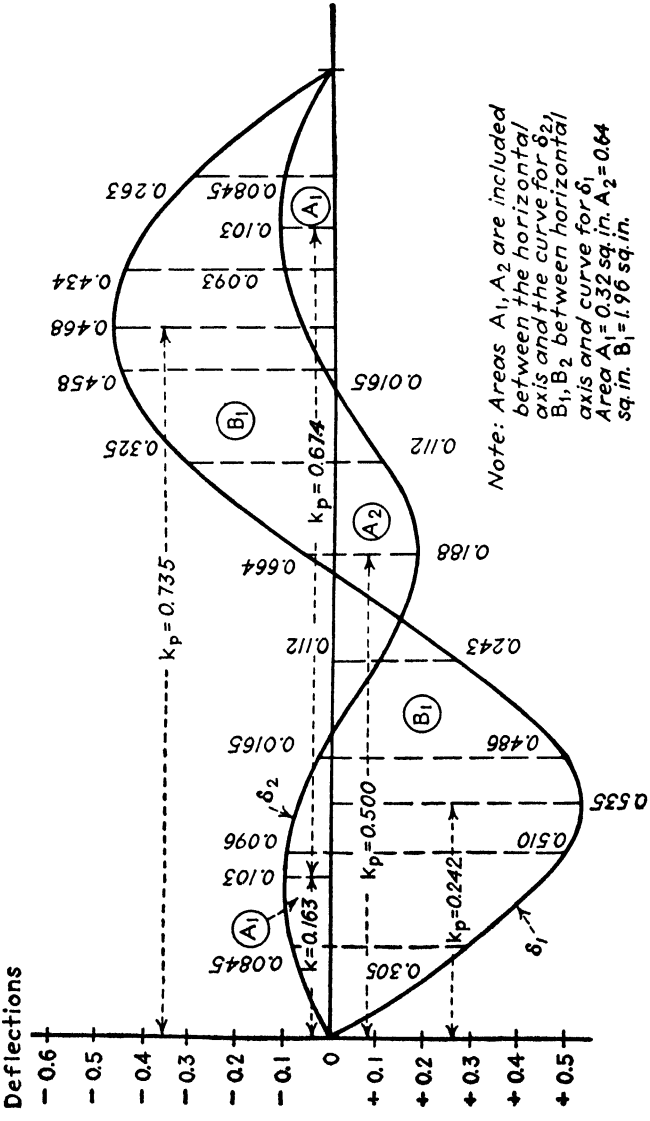
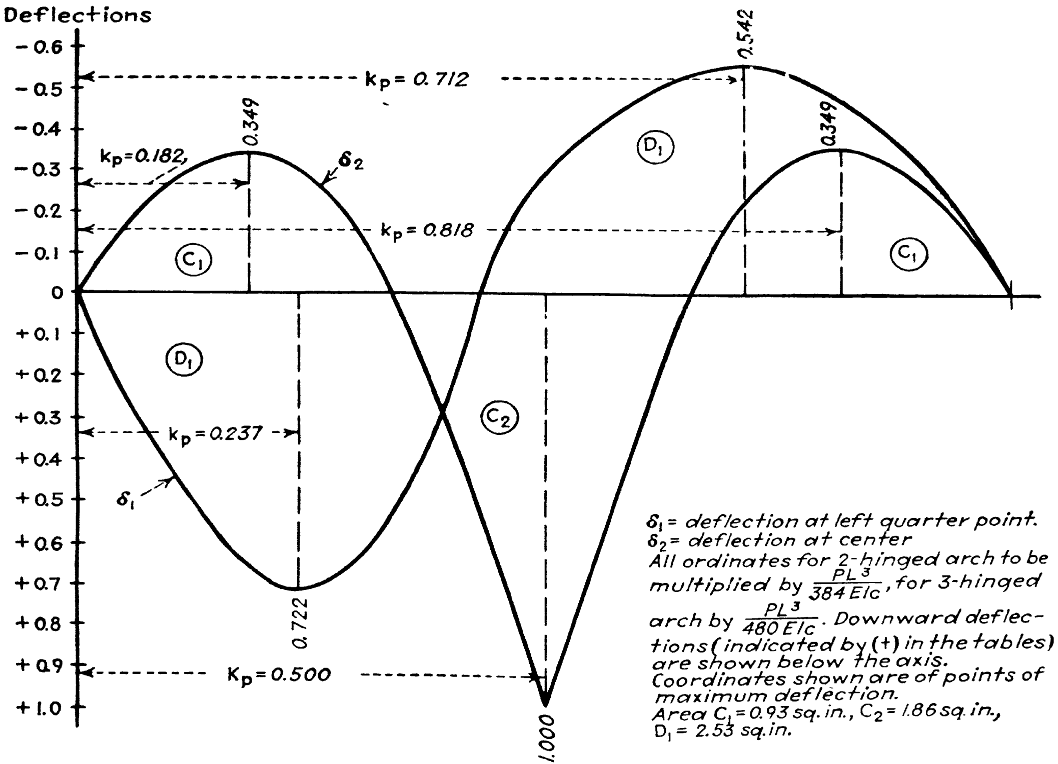


FIG. 3.—Influence line for deflection of a two-hinged parabolic arch due to curvature. See also Fig. 4.



δ_1 = deflection at left quarter point.
δ_2 = deflection at center
All ordinates for 2-hinged arch to be multiplied by $\frac{PL^3}{384EIc}$, for 3-hinged arch by $\frac{PL^3}{480EIc}$. Downward deflections (indicated by (+) in the tables) are shown below the axis.
Coordinates shown are of points of maximum deflection.
Area C_1 = 0.93 sq. in., C_2 = 1.86 sq. in., D_1 = 2.53 sq. in.

FIG. 4.—Influence line for deflection of a three-hinged parabolic arch due to curvature. See also Fig. 3.

from the influence lines corresponding to tabular values in Fig. 3 or 4 or from similar influence lines for arches of other shapes by multiplying the uniform load per linear unit by the area contained between the influence line and the horizontal axis in the loaded portion of the beam. The deflection at either quarter point or crown due to a system of concentrated loads in any given position may be found by multiplying the ordinates of the proper influence line under the respective loads by the values of the loads.

The preceding considerations hold for a uniform load applied directly to the arch rib. For a uniform load applied to the floor system, the deflections are best found by considering the panel loads as a series of concentrated loads applied to the arch rib at the corresponding panel points, multiplying each panel load by the corresponding influence-line ordinates, and summing up the results.

To construct the deflection influence line for the case of loads applied at panel points, proceed as follows:

Divide the base line L into as many equal parts as there are panels in the floor system; at each point of division erect a perpendicular, and connect the points of intersection of these perpendiculars and the influence line as given by the tables corresponding to Fig. 3 or 4, as the case may be, by a series of straight lines. The figures given by these straight lines will be the influence lines for loads applied at the floor system. These lines may be used in the usual manner to determine the position of live loads for maximum deflection.

It will be observed that for a girder divided into panels and under full loading, the area above the axis is no longer equal to the area below the axis; therefore, the deflection at any point for a uniform load applied to the floor system over the entire span will not be zero but will have a value depending upon the algebraic sum of the areas included between the original curve and the series of straight lines. The deflection in this case is small and may usually be neglected unless the number of panels is small; it may be computed, if desired, by multiplying the panel load by the algebraic sum of the ordinates to the influence line at the panel points.

68. Deflection of a Two-hinged Arch Rib Caused by Change in Axial Length of Rib. *Case B.*—For this case the computation of the deflections may be divided into two distinct operations:

1. The determination of the deflection due to change in *length* of the rib, one end being assumed free to move horizontally but not vertically, *i.e.*, to be on rollers.

2. The determination of the deflection due to change in *shape* of the rib due to returning to its original position, the end previously considered as free to move horizontally.

In considering these deflections, it should be observed that the change in length of the rib due to stress may be assumed with little error as entirely due to the axial forces caused by the flexure of the rib, the length of the latter being assumed as constant in determining these forces, *i.e.*, the effect upon its length of the secondary forces due to the axial thrust being neglected in the computations.

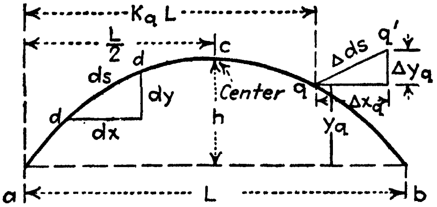


FIG. 5.

The deflection due to these two causes will now be considered separately.

1. *Effect of Change of Length of Rib.*—Consider the curved bar *acb* shown in Fig. 5 with hinges at *a* and *b*, and for the time being assume the bar to be fixed at *a* both in position and in direction but entirely unrestrained at *b*. Consider, now, the movement at any point *q* of the bar, resulting from an increase in length Δds of the portion *ds* of the bar due to any cause whatsoever.

The effect of this change in length will be to cause point *q* to move the distance Δds to *q'* in a direction parallel to *dd*. The two components of this movement will have the following value:

$$\Delta y_q = -\Delta ds\frac{dy}{ds} \qquad \Delta x_q = \Delta ds\frac{dx}{ds}$$

Plus signs in preceding expressions correspond to deflections downward and to right.

The corresponding components due to the total change in length in the bar through its length between *a* and *q* will be given by the following expressions:

$$\delta y_q = -\int_0^{y_q}\left(\frac{\Delta ds}{ds}\right)dy; \qquad \delta x_q = \int_0^{k_qL}\left(\frac{\Delta ds}{ds}\right)dx$$

If we let f = fibre stress, positive when tension, we may put

$$\Delta ds = \frac{f}{E}ds$$

Therefore

$$\delta y_q = -\int_0^{y_q}\frac{f}{E}dy \tag{17}$$

$$\delta x_q = \int_0^{k_qL}\frac{f}{E}dx \tag{18}$$

The foregoing equations are often not integrable; in such cases we may substitute the following:

$$\delta y_q = -\sum_0^{y}\frac{f}{E}\Delta y \tag{19}$$

$$\delta x_q = \sum_0^{x}\frac{f}{E}\Delta x \tag{20}$$

If f is constant throughout the portion of the rib between left hinge and q, and if E is also constant, Eqs. (17) and (18) become

$$\delta y_q = -\frac{f}{E}y_q \tag{21}$$

$$\delta x_q = \frac{f}{E}x_q \tag{22}$$

If fibre stress is compression, as is usually the case, preceding value of δy_q is positive, showing downward deflection, and that of δx_q is negative, showing deflection to left.

In many cases the average value of f may be used with sufficient accuracy in Eqs. (21) and (22).

If these equations are applied to a straight bar making an angle α with the horizontal and of length L, we obtain the following expressions:

$$\delta y_q = -\frac{f}{E}L\sin\alpha; \qquad \delta x_q = \frac{f}{E}L\cos\alpha$$

which are obviously correct for this case.

Up to this point we have considered the b end of the rib to be unrestrained both vertically and horizontally. To determine the vertical deflection of any point *q*, if end *b* is free to move horizontally but not vertically and end *a* is fixed in position and direction, the vertical deflection of *q* as given in the foregoing formulas must be corrected by the amount that point *q* would move if the restraint in direction of end *a* were removed, and the rib swung about point *a* as a pivot until point *b* reaches its correct elevation.

The vertical deflection of point *b* may be determined by either Eq. (17) or Eq. (19), applying the equation first to the left half of the arch to get the vertical deflection of the crown and then to the right half of the arch with the crown as origin to get the vertical deflection of *b* with respect to the crown and combining the results. Equation (17) may then be expressed as follows: If we let f_L = fibre stress in left half of rib and f_R = fibre stress in right half of rib,

$$\delta y_b = -\int_0^h \frac{f_L}{E} dy + \int_0^h \frac{f_R}{E} dy$$

The correction for rotation will then be given by the following equation, which will have a positive value indicating downward deflection if δy_b is negative, *i.e.*, upward:

$$\delta y_q = -k_q(\delta y_b) \qquad (23)$$

In determining this correction, it should be observed that if the changes in length of a symmetrical arch rib are symmetrical with respect to its crown, point *b* will have no vertical deflection with respect to point *a* even if entirely unrestrained, since the vertical movement of point *b* due to change in one half of the rib is balanced by its movement due to change in other half of rib.[1]

2. *Deflection due to Change in Shape of Rib.*—To determine the deflection due to change in shape of rib caused by returning to its original position the end previously considered free to move horizontally, proceed as follows:

a. Compute the horizontal movement δx_b of point *b* by application of either Eq. (18), (20), or (22).

---

[1] The deflection due to temperature changes may be determined from the formulas of this article provided the quantity f/E be replaced by ϵt,
where ϵ = coefficient of linear expansion.
t = change in temperature in degrees, a rise in temperature being considered positive.
The deflection due to shrinkage may be computed by proper substitution.

b. Compute the horizontal force H required to cause an equal movement in the opposite direction, using the following formula (see Appendix D):

$$H = \frac{\delta x_b}{\int \frac{y^2 dx}{EI_c}} \tag{24}$$

Positive value for H indicates H acting inward, *i.e.*, a thrust. δx_b is positive to right.

For a parabolic arch, Eq. (24) becomes from Eq. (5), Chap. V, with nomenclature changed to correspond to that of this chapter,

$$H = \frac{15}{8} \frac{\delta x_b E I_c}{h^2 L} \tag{25}$$

c. Determine the vertical deflection δ_{vq} of point q due to force H applied horizontally at b, assuming the arch free to move horizontally at b, by placing P and H_q equal to zero and H_p equal to H in Eq. (6). The result thus obtained is as follows:

$$\delta_{vq} = \frac{Hhk_qL^2}{3EI_c}(2k_q^2 - k_q^3 - 1) \tag{26}$$

This gives upward deflection if H is a thrust, since $2k^2 - k^3 - 1$ is always negative.

Substituting for H its value obtained from Eq. (25) gives the following for a parabolic arch:

$$\begin{aligned} \delta_{vq} &= \frac{15}{8} \cdot \frac{\delta x_b E I_c}{h^2 L} \cdot \frac{hk_qL^2}{3EI_c}(2k_q^2 - k_q^3 - 1) \\ &= \frac{5}{8} \frac{k_q L \delta x_b}{h}(2k_q^2 - k_q^3 - 1) \end{aligned} \tag{27}$$

For a condition where f may be considered constant,

$$\delta x_b = \frac{f}{E}L \text{ [Eq. (22)]}$$

Therefore, for this case,

$$\delta_{vq} = +\frac{5}{8} \cdot \frac{fL^2}{Eh} \cdot k_q(2k_q^2 - k_q^3 - 1) \tag{28}$$

The foregoing expression gives the *downward* deflection of any point q resulting from change in shape of the arch rib due to an average axial force of f lb. per square inch and will have a positive value, in case the force is compression, provided the usual convention of designating compression as negative is followed, since the last term is always negative.

To obtain the total vertical deflection for a parabolic arch due to change in axial length of rib, the value of δy_q as given by Eqs. (17) and (23) should be added to that given by Eq. (27), resulting in the following expression:

$$\delta_{yq} = -\int_0^{y_q} \frac{f}{E} dy - k_q \int_0^h \frac{f_R - f_L}{E} dy + \frac{5}{8} \frac{k_q L \delta x_b}{h} k_q (2k_q^2 - k_q^3 - 1) \qquad (29)$$

For case where f may be assumed as constant throughout the rib, the total deflection will be as follows:

$$\delta_{yq} = -\frac{f}{E} y_q + \frac{5}{8} \frac{fL^2}{Eh} k_q (2k_q^2 - k_q^3 - 1) \qquad (30)$$

If f is compression and therefore negative, both terms in Eq. (30) will have positive values, showing downward deflection.

For centre of arch,

$$y_q = h \qquad \text{and} \qquad k_q = \tfrac{1}{2}$$

Hence if f is considered constant,

$$\delta y_c = -\frac{f}{E} h + \frac{5}{16} \frac{fL^2}{Eh} \left(-\frac{5}{8} \right) \qquad (30a)$$

The deflection due to a horizontal movement of the abutments may be determined from Eqs. (24) and (26) for the general case and Eq. (27) for the parabolic arch, provided the quantity δx_b is replaced by the total relative horizontal displacement of the abutments, an increase in span being considered as negative.

69. Vertical Deflection of Any Point on a Symmetrical Three-hinged Arch Rib Caused by Changes in Axial Length of the Rib. *Case B.*—Let Fig. 6 represent such a rib. Let the problem be the determination of the vertical deflection of any point q, resulting from a change in the axial length of the rib.

This deflection may be conveniently determined by dividing the work into the following steps:

1. Assume each half rib to be independent of the other, and determine by application of Eqs. (17) and (18) the vertical and

horizontal deflections of point c, assuming each half rib to be fixed against rotation at its end hinge.

2. Determine the vertical movement of point c due to the rotation of each half rib about its respective end hinge until the points c of each half rib meet.

3. Determine deflection of q due to foregoing rotation by multiplying vertical deflection of c during rotation of the half rib containing point q by ratio $2x_q/L$.

4. Assume each half rib to be independent of the other, and determine by application of Eq. (17) the vertical deflection of point q, assuming each rib to be fixed against rotation at its end hinge.

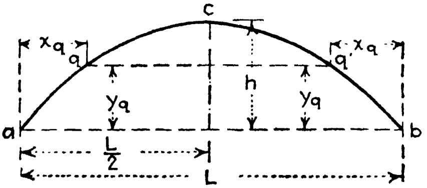


FIG. 6.

5. Combine the vertical deflections of point q obtained in third and fourth steps to give the total vertical deflection of point q.

Now let f = fibre stress in left rib, negative when compression.
f' = fibre stress in right rib, negative when compression.

Then the values in the various steps may be determined as follows:

Step 1.—Let δy_{cL}, δx_{cL}, δy_{cR}, and δx_{cR} = vertical and horizontal deflections, respectively, of point c on each half rib, assuming ribs disconnected and fixed at each end; positive when downward and to right respectively.

Then

$$\delta y_{cL} = -\int_0^h \frac{f}{E}dy; \qquad \delta y_{cR} = -\int_0^h \frac{f'}{E}dy; \qquad \delta x_{cL} = \int_0^{\frac{L}{2}} \frac{f}{E}dx;$$

$$\delta x_{cR} = \int_0^{\frac{L}{2}} \frac{f'}{E}dx$$

Step 2.—Assume both ribs in compression.

In Fig. 7, let δy_{cL}, δy_{cR}, δx_{cL}, and δx_{cR} represent, respectively, the movements of point c, assuming each rib to be independent of the other and fixed at abutment ends. Under these conditions, point c of left rib moves to c'_L, and point c of right rib to c'_R. When the ribs are rotated, each about its end hinge, c'_L and c'_R meet at c''. Since the deflections are small compared with the rib lengths, each of these points may be assumed, when the ribs are rotated, to move on a normal to the rib, these normals being c'_Lc'' and c'_Rc''. The position of c'' may now be determined by writing the equations of these normals and finding their point of intersection. Similar equations may be deduced for the case where both ribs increase in length owing to temperature increases or to tension or where both decrease because of temperature changes or shrinkage and compression or where one rib decreases in length and the other increases.

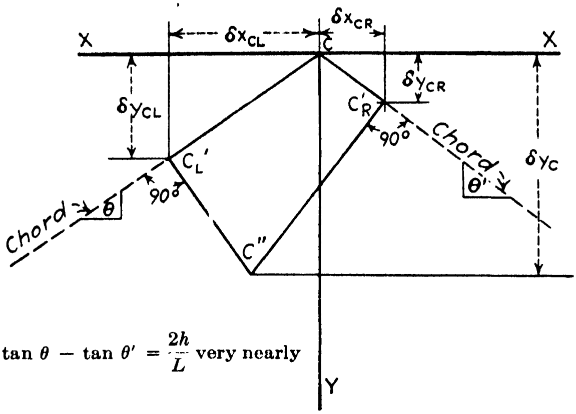


FIG. 7.

The procedure for the case under consideration (both ribs in compression) is as follows:

Assuming x positive when to left of centre axis and y positive downward, and writing the equations of the normals c'_Lc'' and c'_Rc'', we obtain

Equation of c'_Lc'':

$$y = \delta y_{cL} + (\delta x_{cL} - x) \cot \theta \tag{31}$$

Equation of c'_Rc'':

$$y = \delta y_{cR} + (\delta x_{cR} + x) \cot \theta' \tag{32}$$

Equating preceding values of y to determine point of intersection gives

$$\delta y_{cL} + \delta x_{cL} \cot \theta - x \cot \theta = \delta y_{cR} + \delta x_{cR} \cot \theta' + x \cot \theta'$$

$$\cot \theta = \cot \theta' = \frac{L}{2h} \text{ very nearly}$$

Therefore

$$x_{c''} = \frac{\delta y_{cL} + (\delta x_{cL} - \delta x_{cR}) \cot \theta - \delta y_{cR}}{2 \cot \theta}$$

Substituting above value of $x_{c''}$ for x in Eq. (31)

$$\begin{aligned}\delta y_c &= \delta y_{cL} + \delta x_{cL} \cot \theta - \frac{\delta y_{cL}}{2} - \left(\frac{\delta x_{cL} - \delta x_{cR}}{2}\right) \cot \theta + \frac{\delta y_{cR}}{2} \\ &= \frac{\delta y_{cL}}{2} + \frac{\delta y_{cR}}{2} + \frac{\delta x_{cL} \cot \theta}{2} + \frac{\delta x_{cR} \cot \theta}{2} \\ &= \frac{\delta y_{cL} + \delta y_{cR}}{2} + (\delta x_{cL} + \delta x_{cR})\frac{\cot \theta}{2}\end{aligned}$$

But

$$\delta x_{cL} = \frac{\delta y_{cL}}{\tan \theta} \qquad \text{and} \qquad \delta x_{cR} = \frac{\delta y_{cR}}{\tan \theta}$$

hence

$$\delta x_{cL} + \delta x_{cR} = \frac{\delta y_{cL} + \delta y_{cR}}{\tan \theta}$$

Therefore

$$\delta y_c = (\delta y_{cL} + \delta y_{cR}) \left(\frac{1}{2} + \frac{\cot \theta}{2 \tan \theta}\right)$$

but

$$\tan \theta = \frac{2h}{L} \qquad \text{and} \qquad \cot \theta = \frac{L}{2h}$$

hence

$$\frac{\cot \theta}{2 \tan \theta} = \frac{L^2}{8h^2}$$

Therefore

$$\delta y_c = (\delta y_{cL} + \delta y_{cR}) \left(\frac{1}{2} + \frac{L^2}{8h^2}\right) \qquad (33)$$

If

$$\delta y_{cL} = \delta y_{cR}$$

i.e., if stress is constant throughout the entire arch, or if change of temperature is constant throughout arch

$$\delta y_c = \delta y_{cL}\left(1 + \frac{L^2}{4h^2}\right) \tag{34}$$

That this is correct for such a case may be proved by the following independent solution.

In Fig. 8, let c = original position of arch crown on left half of arch and c' and c'' correspond to new position of c, as in previous case. Now, if arch is symmetrical and both halves change equally in length, as will be the case if the average stress throughout each half rib is constant, or if the temperature change is uniform throughout the arch, c'' will lie on a vertical through c. Also, if the left half is assumed fixed at left hinge and free to move at centre, point c will move to point c' which will be on the chord connecting left and centre hinges, since the relation between the vertical and horizontal movements of point c due to change in length of arch rib is along the rib itself.

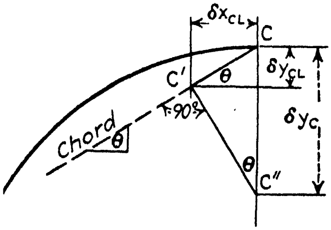


FIG. 8.

If motion of c' due to rotation is assumed to be at right angles to chord, as in previous case, then

$$\delta y_c = \delta y_{cL} + \delta x_{cL} \cot \theta$$

but

$$\delta x_{cL} = \frac{\delta y_{cL}}{\tan \theta}$$

Therefore

$$\delta y_c = \delta y_{cL}\left(1 + \frac{\cot \theta}{\tan \theta}\right) = \delta y_{cL}\left(1 + \frac{L/2h}{2h/L}\right)$$

$$= \delta y_{cL}\left(1 + \frac{L^2}{4h^2}\right)$$

which equals the value given by Eq. (34).

For a numerical case, consider $f = 5800$* lb. compression; and E, 29,000,000. Also, $L = 400$ ft. and $h = 40$ ft. Then

$$\delta y_{cL} = \frac{h}{5000}$$

and

$$\delta y_c = \frac{1}{5000}\left(40 + \frac{400 \times 400}{160}\right) = \frac{1040'}{5000} = \text{downward deflection}$$

If $L = 400$, and $h = 80$,

$$\delta y_c = \frac{1}{5000}\left(80 + \frac{400 \times 400}{320}\right) = \frac{580'}{5000} = \text{downward deflection}$$

It is evident from the foregoing numerical examples that the actual deflection of point c will be so small compared with the arch span that the error in assuming point c' to move in a normal to the chord is insignificant.

It should be observed that the actual deflection may be readily found graphically by plotting the position of c' and c to any desired scale, erecting normals to the ribs at these points, and determining the point of intersection which will be the correct position of c'' to the scale chosen. This is identical with the Williot-diagram method.

Step 3.—During the rotation of the left half rib the vertical deflection of point c equals δy_c minus δy_{cL}, hence, from Eq. (33):

$$(\delta y_c - \delta y_{cL}) = \delta y_{cR}\left(\frac{1}{2} + \frac{L^2}{8h^2}\right) + \delta y_{cL}\left(\frac{L^2}{8h^2} - \frac{1}{2}\right) \tag{35}$$

Also, during the rotation of the right half rib the vertical deflection of point c equals δy_c minus δy_{cR}:

$$(\delta y_c - \delta y_{cR}) = \delta y_{cR}\left(\frac{L^2}{8h^2} - \frac{1}{2}\right) + \delta y_{cL}\left(\frac{1}{2} + \frac{L^2}{8h^2}\right) \tag{36}$$

If

$$\delta y_{cL} = \delta y_{cR}$$

i.e., if stress is constant throughout entire arch, or if change in temperature is constant throughout arch,

$$(\delta y_c - \delta y_{cL}) = (\delta y_c - \delta y_{cR}) = \delta y_{cL}\left(\frac{L^2}{4h^2}\right) \tag{37}$$

---

* The average fibre stress in an arch is usually much less than the maximum fibre stress due to bending and direct stress.

The vertical deflection $\delta y'_q$ of point q due to rotation determined under step 2 equals the product of $2x_q$ and either $(\delta y_c - \delta y_{cL})$ or $(\delta y_c - \delta y_{cR})$ depending upon whether point q is contained in left or right half rib, respectively.

If stress is constant throughout entire arch, or if change in temperature is constant throughout arch,

$$\delta y'_q = \frac{2x_q}{L}(\delta y_{cL})\frac{L^2}{4h^2} = \frac{x_q L}{2h^2}\delta y_{cL} \tag{38}$$

Step 4.—Let $\delta y''_q$ = vertical deflection of point q, assuming each half rib to be independent of the other and to be fixed against rotation at its end hinge. Taking as an origin the end hinge of the half rib containing point q, and applying Eq. (17),

$$\delta y''_q = -\int_0^{y_q} \frac{f}{E} dy \tag{39}$$

From Eq. (39), if f is constant,

$$\delta y''_q = -\frac{f}{E} y_q \tag{40}$$

Step 5.—Let δ_{vq} = total vertical deflection of point q. Then δ_q equals sum of deflections found in steps 3 and 4,

$$\delta_{vq} = \delta y'_q + \delta y''_q \tag{41}$$

If f is constant throughout entire arch,

$$\delta_{vq} = \frac{x_q L}{2h^2}\delta y_{cL} - \frac{f}{E} y_q$$

But

$$\delta y_{cL} = -\frac{f}{E} h$$

Hence,

$$\delta_{vq} = -\frac{f}{E}\left(y_q + \frac{x_q L}{2h}\right) \tag{42}$$

If f is compression and consequently negative, preceding expression will come out "positive," showing downward deflection.

If point q is at centre,

$$\delta_{vc} = -\frac{f}{E}\left(h + \frac{L^2}{4h}\right) \tag{43}$$

which agrees with value given by Eq. (34), as it should.

70. Comparison of Deflection Due to Change of Axial Length in Two-hinged and Three-hinged Arch Ribs.—Let δ_3 and δ_2 represent, respectively, the centre deflections due to axial stress in a three-hinged and a two-hinged symmetrical-arch rib due to a constant compressive unit stress f throughout length of each rib. Hence

$$\frac{\delta_3}{\delta_2} = \frac{\text{value given by Eq. (43)}}{\text{value given by Eq. (30}a)}$$

$$= \frac{\dfrac{f}{E}\left(h + \dfrac{L^2}{4h}\right)}{\dfrac{f}{E}\left(h + \dfrac{25}{32} \times \dfrac{L^2}{4h}\right)}$$

showing that the two values are nearly identical, the three-hinged arch having somewhat the greater deflection.

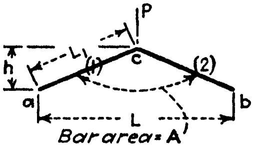


FIG. 9.

71. Test for Special Case.—Equation (43) may readily be verified for the deflection due to a vertical load applied at the centre of a structure like that shown in Fig. 9.

The total stress in bars 1 and 2 of the structure is constant throughout the length of each bar and equals $-\dfrac{PL_1}{2h}$.

The unit stress in each bar $= -\dfrac{PL_1}{2hA}$.

The change in length of each bar $= -\dfrac{PL_1}{2h}\dfrac{L_1}{EA}$.

The vertical deflection of point c by the ordinary method of truss deflection $= 2(L_1/2h)(PL_1^2/2hEA) = PL_1^3/2h^2EA$

From Eq. (43) we obtain by substituting for f its value as found above

$$\delta_{yc} = \frac{PL_1}{2hEA}\left(h + \frac{L^2}{4h}\right)$$

But

$$L_1^2 = h^2 + \frac{L^2}{4} = h\left(h + \frac{L^2}{4h}\right)$$

Therefore

$$h + \frac{L^2}{4h} = \frac{L_1^2}{h}$$

Therefore $\delta_{yc} = PL_1^3/2h^2EA$, which agrees with the value previously obtained.

72. Deflection Due to Temperature Change.—To find the deflection of point q resulting from changes in temperature, substitute for f/E in Eq. (42) the product of the change in temperature and the coefficient of expansion (see Art. 68).

Problems

1. *a.* Determine the maximum downward vertical deflection at centre for both two-hinged and three-hinged parabolic-arch ribs due to following loading:

Dead load = 1614 lb. per lineal foot
Live load = 1323 lb. per lineal foot

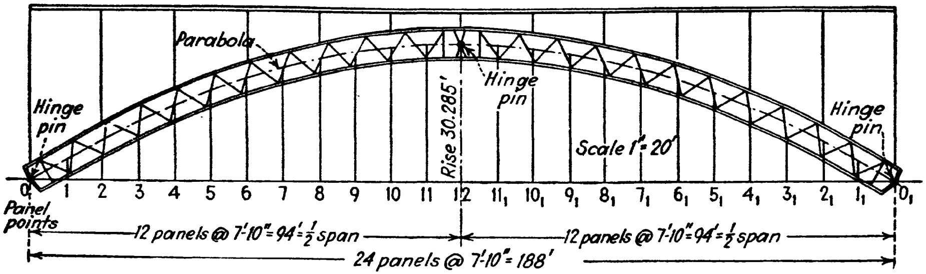
Cross section of arch rib same as that in the illustrative problem in Art. 36.

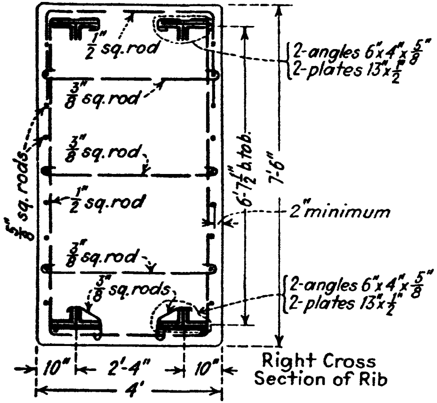
Span = 145 ft. 0 in.; rise = 21 ft. 0 in.; E = 29,000,000 lb. per square inch.

b. Corresponding deflections for temperature drop of 60°F.:

$$\epsilon = 0.0000067$$

2. See next page.





Right Cross Section of Rib

Loads to be used:

Weight of floor system, etc., above ribs	4000 lb./ft. of span
" " steel rib	800 " " "
" " concrete in rib and bracing	5,500 " " "
Live load	4,300 " " "

Compute deflection for the crown and the one-quarter point

Due to	Hinges in arch	Carried by
(1) Weight of steel rib	3	Steel
(2) Weight of concrete in rib and bracing	3	Steel
(3) Weight of floor system, etc., above rib	3	Steel and concrete
(4) Live load	2	Steel and concrete
(5) Drop in temp. of 40°F.	2	Steel and concrete

Assume average axial thrust in all cases equal to $H\sqrt{1+5.26\left(\frac{H}{L}\right)^2}$

Mix of concrete 1:2:4
Value of $E_c = 2{,}500{,}000$ lb./sq.in.
Use $n = 12$

Prob. 2.

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CHAPTER VIII

FIXED-ENDED ARCH

73. Preliminary Determination of Thickness of Arch Rib. Fixed-ended Arch.—With the span and rise determined by the method outlined in Chap. II, the thickness at crown must next be assumed. In deciding upon this thickness, existing designs may be used as a basis, or its value may be computed by an empirical formula such as the following published by F. W. Weld in *Engineering Record*, Nov. 4, 1905:

$$d_c = \sqrt{L} + \frac{L}{10} + \frac{w_L}{200} + \frac{w_c}{400}$$

where d_c = crown thickness of barrel arch in inches.
L = clear span in feet.
w_L = live load in pounds per square foot, 50 per cent to be added for impact on railroad bridge.
w_c = weight of fill over crown in pounds per square foot.

After deciding upon the crown thickness, the next step is to make an assumption as to the springing thickness. A reasonable assumption is to make this thickness from one and one-half to three times the crown thickness, depending upon the flatness of the arch, the larger value corresponding to the flatter arch.

Between the crown and skewback the arch rib should be gradually increased in thickness. Whitney (see reference at end of chapter) gives the following equation for determining the thickness of the rib at intermediate points:

Let d_c = thickness at crown.
d_s = thickness at springing.
d_k = thickness at distance kL from crown.
I_c = moment of inertia at crown.
I_s = moment of inertia at springing.

Other terms are as shown in Fig. 1.

Then $d_k = d_c c\sqrt[6]{1 + \tan^2 \theta_k}$, in which

$$c = \frac{1}{\sqrt[3]{1 - 2(1 - n)k}}$$

and

$$n = \frac{I_c}{I_s \cos \theta_s}$$

The use of preceding expression requires considerable computing; it is possibly equally satisfactory to determine the shape of the rib between crown and springing by eye, bearing in mind that the value of the moment which is the controlling factor in the design is about the same at the quarter point as at the crown but increases rapidly from quarter point to springing.

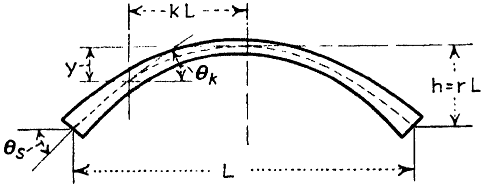

FIG. 1.—θ_K = angle between tangent to arch axis and horizontal.

74. Shape of Arch Axis. Fixed-ended Arch.—With thickness at crown and springing assumed, the shape of the arch axis may now be investigated. Various equations for the axes of fixed-ended masonry arches have been developed in recent years, of which those by Cochrane and by Whitney (see references at end of chapter) have received favorable consideration by engineers and will now be given:

Cochrane Equations for Arch Axis (see Fig. 1).

Filled-spandrel arch:

$$y = \frac{4rL}{1 + 3r}(k^2 + 24k^5 r).$$

$$\tan \theta_s = \frac{4r}{1 + 3r}(1 + 7.5r).$$

Open-spandrel arch:

$$y = \frac{8rL}{6 + 5r}(3k^2 + 10k^4 r).$$

$$\tan \theta_s = \frac{8r}{6 + 5r}(3 + 5r).$$

Whitney Equations for Arch Axis (see Fig. 1).

Open- or closed-spandrel arch:

$$y = \frac{h}{g - 1}(\cosh 2km - 1).$$

$$\tan \theta_k = \frac{2hm}{L(g - 1)} \sinh (2km).$$

In foregoing expression,

$g = w_s/w_c.$

$m = \cosh^{-1} g.$

w_s = dead load per unit length of span at springing.

w_c = dead load per unit length of span at crown.

If the arch is of the open-spandrel type with the load carried to the arch rib by columns or transverse walls, the values of w_c and w_s may be obtained by considering the panel loads to be distributed over a panel length and assuming the column construction to extend beyond the abutment a distance equal to the distance between the spandrel column and the adjoining column.

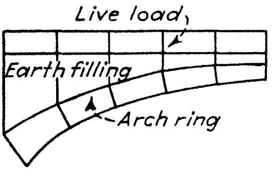

FIG. 2.

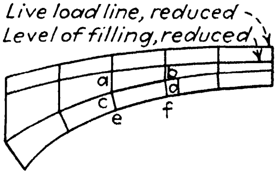

FIG. 3.

75. Outer Forces. Fixed-ended Arch.—After the arch ring is assumed, the loads must be determined. To accomplish this, divide the arch ring into sections of reasonable length, as shown in Fig. 2, and compute the magnitude and point of application of the load acting on each section, including the weight of the arch itself.

The dead weight is usually reduced to units corresponding to the weight of the arch ring as shown in Fig. 3; *e.g.*, if the filling weighs 120 lb. per cubic foot, and the masonry of the arch ring 150 lb., a 12-ft. height of filling would be plotted above the arch ring, as $12 \times {}^{120}\!/_{150} = 9.6$ ft. The areas and location of the centres of gravity of the figures *abcd* and *cefd* (Fig. 3) may next be determined. Both of these areas may be considered as trapezoids. The points of application of the resultant of the weights represented by trapezoids *abcd* and *cdef* must then be determined, either graphically or analytically.[1]

It is sometimes simpler to determine analytically the position of the point of application of the resultant of the weight of the filling and the arch ring rather than to reduce to common units. Impact is generally disregarded in earth-covered masonry arches owing to the deadening effect of the filling.

With respect to the position of the live load, it may be said that full live load over the entire arch does not give the worst condition for stability. The only thoroughly satisfactory method of determining the position of the live loads is by the construction of influence lines, as illustrated later in the chapter. It may be noted that a uniform live load extending from left end approximately three-eighths of the distance to the centre gives maximum negative moment at left skewback and maximum positive moment at left quarter point. A load over remainder of span consequently gives maximum moments of opposite character at these sections. For maximum positive crown moment the live load should cover approximately the middle quarter of the span, and for maximum negative live moment three-eighths of the span from each end, *i.e.*, three-fourths of the span in all.

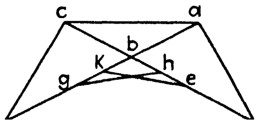

FIG. 4.

With the arch ring assumed, and the outer forces determined, the next step is the analysis of the stresses in the assumed arch rib. The expressions developed in the following articles have been found by the author to be simple to understand and apply and are recommended for use.

76. General.—An arch with fixed ends, *i.e.*, fixed in position and direction, is indeterminate to the third degree with respect to the outer forces. The three unknowns which cannot be determined by statics may be taken either as three of the reaction functions or as a moment, shear and thrust acting at the crown or at any other convenient section of the arch.

The equations developed in this chapter for the unknowns are based upon the following assumptions in addition to those of the ordinary beam formula:

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[1] To determine the centre of gravity of a trapezoid, the method illustrated by Fig. 4 is often convenient. Lay off *fg* = *ab* and *de* = *cb*. Bisect *bg* and *be*. The centre of gravity is at the intersection of *ek* and *gh*.

1. The ratio of the depth of rib to the radius of curvature is such that the effect of curvature on the distribution of the stress over the cross section may be neglected.
2. The axial thrust is constant throughout the arch and in the case of symmetrical arches carrying vertical loads is approximately equal to the horizontal component of the crown thrust.
3. The effect of shearing distortion is neglected.

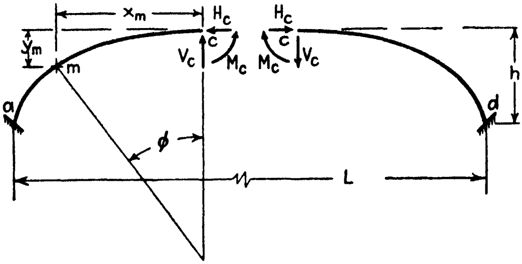
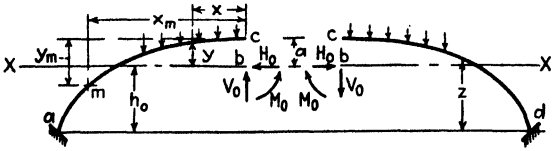


FIG. 5.

77. Transfer of Crown Forces to Another Position.—The moment, thrust, and shear acting at the crown of any arch can be replaced if desired by equivalent forces acting at any other convenient location without changing the conditions of equilibrium of the arch; *e.g.*, M_c, H_c, and V_c shown in Fig. 5 may be replaced



Let y be positive upward with reference to axis X-X
" x " " in either direction from center

FIG. 6.

by M_0, H_0, and V_0 acting about an axis passing through point b in Fig. 6.

In order that these forces and moments may have the same effect upon the arch as the original unknowns, the following relations will have to exist:

$$M_0 = M_c + H_0 a; \qquad H_0 = H_c; \qquad V_0 = V_c$$

That the resulting moment, shear, and thrust upon any section of the arch is unaffected by this transformation may readily be shown by considering its effect upon any section such as that at m.

The moment at this section due to M_c, V_c, and H_c =

$$M_c + V_c x_m + H_c y_m.$$

The moment at the same section due to M_0, V_0, and H_0 =

$$M_0 + V_0 x_m + H_0(y_m - a).$$

But $M_0 = M_c + H_c a$, $V_0 = V_c$, and $H_0 = H_c$
Substituting these values gives the moment =

$$M_c + V_c x_m + H_c y_m$$

or the same equation as that previously determined.
The shear $= V_c \cos \phi + H_c \sin \phi = V_0 \cos \varphi + H_0 \sin \varphi$

The thrust $= H_c \cos \phi - V_c \sin \phi = H_0 \cos \varphi - V_0 \sin \varphi$.

78. Method of Solution for a Symmetrical Arch.—In the solution that follows, the unknowns will be taken as M_0, H_0, and V_0 acting at point *b*, as shown in Fig. 6, in which the outer forces applied to the arch are also indicated. Point *b* is located on a vertical line through the mid-span point on the arch axis at a distance h_0 above the axis passing through the end hinges such that $\int \frac{yds}{EI} = 0$. In other words, point *b* lies on an axis passing through the centroid of the vertical projection of the arch rib. If *E* and *I* are constant, it passes through the centroid of the vertical projection of the arch axis.

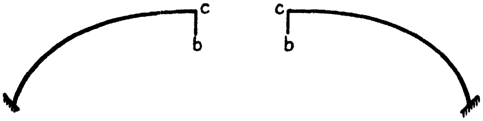


FIG. 7.

Assume point *b* connected to point *c* on each half of the arch by a rigid bar, as shown in Fig. 7, and equate:

1. The vertical deflections of point *b* on each half of arch.
2. The horizontal deflections of point *b* on each half of arch.
3. The slope of the arch at *b* on each half of the arch.

79. Fundamental Equations for a Symmetrical Arch.

Let M = actual bending moment in foot-pounds, or other appropriate units, on a normal section of the arch at a distance x from the origin.

m_0 = numerical coefficient of the moment on a normal section of the arch at a distance x from the origin due to load of unity, the latter acting at point b and in the direction of the desired deflection, *i.e.*, either vertical or horizontal.

m_1 = numerical coefficient of the moment on a normal section of the arch at a distance x from the origin due to unit moment acting at c.

m_R and m_L = bending moment on right and left half of arch, respectively, due to applied loads, considering each half of arch to act as a cantilever fixed at right and left ends, respectively.

T = axial force applied to member at any normal section.

The deflection and the change of slope at any point due to bending are given, respectively, by the following equations:[1]

$$\text{Deflection} = \int \frac{Mm_0 ds}{EI}$$

$$\text{Change of slope} = \int \frac{Mm_1 ds}{EI}$$

The horizontal and vertical deflections of point b due to change in length of the arch caused by axial thrust T are as follows:

$$\text{Horizontal} = \int_0^{\frac{L}{2}} \frac{Tds}{AE} \cdot \frac{dx}{ds}$$

$$\text{Vertical} = \int_0^{h} \frac{Tds}{AE} \cdot \frac{dy}{ds}$$

The change of slope at crown due to the axial thrust is extremely small and may be neglected.

Now,

$$M = (M_0 - H_0 y + V_0 x - m_L) \text{ for left half of arch}$$

and

$$M = (M_0 - H_0 y - V_0 x - m_R) \text{ for right half of arch}$$

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[1] See Theory of Structures, 3d ed., Spofford.

Let δ_{Lh} = horizontal deflection to left, of point b on left half of arch.

δ_{Rh} = horizontal deflection to left, of point b on right half of arch.

δ_{Lv} = vertical deflection downward, of point b on left half of arch.

δ_{Rv} = vertical deflection downward, of point b on right half of arch.

α_L = change in slope clockwise, of point c on left half of arch.

α_R = change in slope clockwise, of point c on right half of arch.

Now,

$$\delta_{Lh} = \delta_{Rh}; \qquad \delta_{Lv} = \delta_{Rv}; \qquad \alpha_L = \alpha_R$$

But

$$\delta_{Lh} = \int_0^{\frac{S}{2}} (M_0 - H_0 y + V_0 x - m_L)\left(-\frac{yds}{EI}\right) + \int_0^{\frac{L}{2}} \frac{Tds}{AE}\frac{dx}{ds} \tag{1}$$

$$\delta_{Rh} = \int_0^{\frac{S}{2}} (M_0 - H_0 y - V_0 x - m_R)\left(\frac{yds}{EI}\right) - \int_0^{\frac{L}{2}} \frac{Tds}{AE}\cdot\frac{dx}{ds} \tag{2}$$

$$\delta_{Lv} = -\int_0^{\frac{S}{2}} (M_0 - H_0 y + V_0 x - m_L)\left(\frac{xds}{EI}\right) + \int_0^{h} \frac{Tds}{AE}\frac{dy}{ds} \tag{3}$$

$$\delta_{Rv} = -\int_0^{\frac{S}{2}} (M_0 - H_0 y - V_0 x - m_R)\left(\frac{xds}{EI}\right) + \int_0^{h} \frac{Tds}{AE}\frac{dy}{ds} \tag{4}$$

$$\alpha_L = -\int_0^{\frac{S}{2}} (M_0 - H_0 y + V_0 x - m_L)\frac{ds}{EI} \tag{5}$$

$$\alpha_R = \int_0^{\frac{S}{2}} (M_0 - H_0 y - V_0 x - m_R)\left(\frac{ds}{EI}\right) \tag{6}$$

But by construction $\int \frac{yds}{EI} = 0$; and M_0, H_0 and V_0 are constants for any given loading.

Assume hereafter that T is constant throughout and equal to H_0. This is somewhat approximate, but investigation of certain bridge arches shows that this assumption is but slightly in error. (See also Arts. 47 and 53.)

By equating the appropriate expressions for deflection and slope, we obtain the following equations for symmetrical arches having a constant value of E. From Eqs. (1) and (2),

$$2H_0\int_0^{\frac{S}{2}}\frac{y^2ds}{I} + \int_0^{\frac{S}{2}} m_L y\frac{ds}{I} + \int_0^{\frac{S}{2}} m_R y\frac{ds}{I} + 2H_0\int_0^{\frac{S}{2}}\frac{\cos\phi\, ds}{A} = 0$$

Therefore

$$H_0 = \frac{-\int_0^{\frac{S}{2}} m_L y\frac{ds}{I} - \int_0^{\frac{S}{2}} m_R y\frac{ds}{I}}{2\int_0^{\frac{S}{2}}\frac{y^2ds}{I} + 2\int_0^{\frac{S}{2}}\frac{\cos\phi\, ds}{A}} \tag{7}$$

From Eqs. (3) and (4),

$$2V_0\int_0^{\frac{S}{2}} x^2\frac{ds}{EI} - \int_0^{\frac{S}{2}} m_L x\frac{ds}{I} + \int_0^{\frac{S}{2}} m_R x\frac{ds}{I} = 0$$

Therefore

$$V_0 = \frac{\int_0^{\frac{S}{2}} m_L x\frac{ds}{I} - \int_0^{\frac{S}{2}} m_R x\frac{ds}{I}}{2\int_0^{\frac{S}{2}} x^2\frac{ds}{I}} \tag{8}$$

From Eqs. (5) and (6),

$$2M_0\int_0^{\frac{S}{2}}\frac{ds}{I} - \int_0^{\frac{S}{2}} m_L\frac{ds}{I} - \int_0^{\frac{S}{2}} m_R\frac{ds}{I} = 0$$

Therefore

$$M_0 = \frac{\int_0^{\frac{S}{2}} m_L\frac{ds}{I} + \int_0^{\frac{S}{2}} m_R\frac{ds}{I}}{2\int_0^{\frac{S}{2}}\frac{ds}{I}} \tag{9}$$

As integration of the foregoing equations is difficult, it is advisable to divide the arch axis into parts of equal length Δs and to replace the integral sign by that of summation giving the following equations for a symmetrical arch. The summations, except for the moment terms, apply to one-half of the arch only. The second term in the denominator is small, and for ordinary arches no serious error will be made in substituting for cos ϕ the value of unity, as has been done in the following equations.

$$H_0 = -\frac{1}{2}\frac{\left[\sum m_L\frac{y}{I} + \sum m_R\frac{y}{I}\right]}{\sum\frac{y^2}{I} + \sum\frac{1}{A}} \tag{10}$$

$$V_0 = \frac{1}{2}\frac{\left[\sum m_L\frac{x}{I} - \sum m_R\frac{x}{I}\right]}{\sum\frac{x^2}{I}} \tag{11}$$

$$M_0 = \frac{1}{2}\frac{\left[\sum\frac{m_L}{I} + \sum\frac{m_R}{I}\right]}{\sum\frac{1}{I}} \tag{12}$$

In analyzing a barrel arch, a 1-ft. slice is usually investigated, in which case I may be replaced by $d^3/12$, and A by d. Then Eqs. (10), (11), and (12) become

$$H_0 = -\frac{1}{2}\frac{\left[\sum m_L\frac{y}{d^3} + \sum m_R\frac{y}{d^3}\right]}{\sum\frac{y^2}{d^3} + \sum\frac{1}{12d}} \tag{10a}$$

$$V_0 = \frac{1}{2}\frac{\sum m_L\frac{x}{d^3} - \sum m_R\frac{x}{d^3}}{\sum\frac{x^2}{d^3}} \tag{11a}$$

$$M_0 = \frac{1}{2}\frac{\sum\frac{m_L}{d^3} + \sum\frac{m_R}{d^3}}{\sum\frac{1}{d^3}} \tag{12a}$$

For the case of an unsymmetrical arch which is unsymmetrical by being on a gradient but is symmetrical about an axis through the crown normal to a line connecting the springing points of the axis, the method used in Art. 48 may be adopted with suitable modifications. This method makes necessary the development of equations for the values of m_L and m_R due to a load parallel to the axis connecting the springing points of the axis, values that can be readily combined with those due to vertical loads.

80. Temperature Stresses for a Symmetrical Arch.—For stresses due to uniform change in temperature throughout the entire arch rib, V_0, m_L, and m_R each equal zero, and the horizontal component of the thrust is constant throughout the length of the arch rib, since there are no vertical loads. Also, the axial thrust at any section $= H_0 \cos \phi$.

Now, let ϵ = coefficient of expansion.

$t°$ = increase in temperature in degrees.

Then for a symmetrical arch symmetrically *heated*,

$$\delta_{Lh} = \int_0^{\frac{S}{2}} (M_0 - H_0 y)\left(-\frac{yds}{EI}\right) + \int_0^{\frac{S}{2}} \frac{H_0 \cos \phi}{AE}\frac{dsdx}{ds} - \frac{\epsilon t° L}{2} = 0$$

But by construction

$$\int \frac{M_0 yds}{EI} = 0$$

Also

$$\frac{dx}{ds} = \cos \phi$$

Therefore

$$H_0 \int_0^{\frac{S}{2}} \frac{y^2 ds}{EI} + H_0 \int_0^{\frac{S}{2}} \frac{\cos^2 \phi ds}{AE} - \frac{\epsilon t° L}{2} = 0$$

Therefore

$$H_0 = \frac{1}{2} \frac{\epsilon t° L}{\int_0^{\frac{S}{2}} \frac{y^2 ds}{EI} + \int_0^{\frac{S}{2}} \frac{\cos^2 \phi ds}{AE}} \tag{13}$$

For decrease in temperature, the value of H_0 would be negative.

Since

$$\int_0^{\frac{S}{2}} \cos^2 \phi ds = \int_0^{\frac{L}{2}} \cos \phi dx$$

we may also write:

$$H_0 = \frac{1}{2}\frac{\epsilon t^\circ L}{\int_0^{\frac{S}{2}} \frac{y^2 ds}{EI} + \int_0^{\frac{L}{2}} \frac{\cos\phi dx}{AE}} \tag{14}$$

Since the second term in the denominator is small, and $\cos\phi$ is approximately equal to unity, this term may, if E and A are constant, be written:

$$\int_0^{\frac{L}{2}} \frac{dx}{AE} = \frac{L}{2AE}$$

In consequence, for the case where E and A are constant:

$$H_0 = \frac{1}{2}\frac{\epsilon t^\circ LE}{\int_0^{\frac{S}{2}} \frac{y^2 ds}{I} + \frac{L}{2A}} \tag{15}$$

81. Effect of Movement of Abutments for a Symmetrical Arch.—The effect of movements of the abutments upon M_0, H_0, and V_0 may be determined as follows. In evaluating this effect the arch will be considered as not subjected to applied loads or change in temperature.

Let Δx_a, Δx_d = horizontal movement of points a and d, respectively, positive to the right.

Δy_a, Δy_d = vertical movements of points a and d, respectively, positive downward.

α_a, α_d = rotation of tangent to axis of arch at points a and d, respectively, positive when clockwise.

h_0 = distance as shown in Fig. 6.

Using the notation and procedure of Art. 79,

$$\delta_{Lh} = \int_0^{\frac{S}{2}} (M_0 - H_0 y + V_0 x)\left(\frac{-y ds}{EI}\right) + \int_0^{\frac{L}{2}} \frac{T ds}{AE}\frac{dx}{ds} - \Delta x_a - h_0 \alpha_a \tag{16}$$

$$\delta_{Rh} = \int_0^{\frac{S}{2}} (M_0 - H_0 y - V_0 x)\left(\frac{y ds}{EI}\right) - \int_0^{\frac{L}{2}} \frac{T ds}{AE}\frac{dx}{ds} - \Delta x_d - h_0 \alpha_d \tag{17}$$

$$\delta_{LV} = -\int_0^{\frac{S}{2}}(M_0 - H_0y + V_0x)\left(\frac{xds}{EI}\right) + \int_0^{h}\frac{Tds}{AE}\frac{dy}{ds} + \Delta y_a + \frac{L}{2}\alpha_a \tag{18}$$

$$\delta_{RV} = -\int_0^{\frac{S}{2}}(M_0 - H_0y - V_0x)\left(\frac{xds}{EI}\right) + \int_0^{h}\frac{Tds}{AE}\frac{dy}{ds} + \Delta y_d - \frac{L}{2}\alpha_d \tag{19}$$

$$\alpha_L = -\int_0^{\frac{S}{2}}(M_0 - H_0y + V_0x)\frac{ds}{EI} + \alpha_a \tag{20}$$

$$\alpha_R = \int_0^{\frac{S}{2}}(M_0 - H_0y - V_0x)\frac{ds}{EI} + \alpha_d \tag{21}$$

Assuming that $T = H_0 \cos \phi$, and noting that

$$\delta_{Lh} = \delta_{Rh}, \qquad \delta_{LV} = \delta_{RV}, \qquad \text{and} \qquad \alpha_L = \alpha_R$$

gives the following results if E is constant. From Eqs. (16) and (17),

$$2H_0\int_0^{\frac{S}{2}}\frac{y^2ds}{EI} + 2H_0\int^{\frac{S}{2}}\frac{\cos^2\phi ds}{AE} = (\Delta x_a - \Delta x_d) + h_0(\alpha_a - \alpha_d)$$

Therefore

$$H_0 = \frac{E(\Delta x_a - \Delta x_d) + Eh_0(\alpha_a - \alpha_d)}{2\int_0^{\frac{S}{2}}\frac{y^2ds}{I} + 2\int_0^{\frac{S}{2}}\frac{\cos^2\phi ds}{A}} \tag{22}$$

From Eqs. (18) and (19),

$$-2V_0\int_0^{\frac{S}{2}}\frac{x^2ds}{EI} = -(\Delta y_a - \Delta y_d) - \frac{L}{2}(\alpha_a + \alpha_d)$$

Therefore

$$V_0 = \frac{E(\Delta y_a - \Delta y_d) + \frac{EL}{2}(\alpha_a + \alpha_d)}{2\int_0^{\frac{S}{2}}\frac{x^2ds}{I}} \tag{23}$$

From Eqs. (20) and (21),

$$-2M_0 \int_0^{\frac{S}{2}} \frac{ds}{EI} = -(\alpha_a - \alpha_d)$$

Therefore

$$M_0 = \frac{E(\alpha_a - \alpha_d)}{2\int_0^{\frac{S}{2}} \frac{ds}{I}} \tag{24}$$

Equations (22), (23), and (24) may be used to evaluate the effect of any given movement of the abutments upon H_0, V_0, and M_0. It should be noted that a horizontal movement affects only H_0, a vertical movement affects only V_0, and a rotation affects all three.

82. Line of Resistance.—It is always advisable to construct a line of resistance for an arch after the values of the various unknowns have been determined. This may be done in a manner similar to that employed in the case of a masonry dam (see Theory of Structures, Spofford). For a symmetrical arch symmetrically loaded, the resultant crown thrust to use in constructing the line of resistance equals H_0, is horizontal, and is located at a distance above the axis XX equal to M_0/H_0.

For unsymmetrical loading, the point of application of the resultant crown thrust may be obtained from the same formula, but the thrust is no longer horizontal, its slope being given by the expression H_0/V_0. Its horizontal component is H_0. With the crown thrust fixed in position and direction, the line of resistance can be readily constructed. It should be kept within the middle third of the arch ring for unreinforced structures but may be allowed to pass outside the middle third for reinforced concrete or steel arches.

83. Distribution of Stress over Cross Section.—The maximum fibre stress in an unreinforced arch may be computed by the methods used in Art. 36. Its value depends upon the eccentricity and magnitude of the thrust at a given section, and several sections may have to be tried to determine the limiting condition. If the arch is of reinforced concrete, the maximum stress upon any section, the resultant upon which passes through the kernel, may be found in a similar manner, the steel being assumed as replaced by an equivalent amount of concrete, *i.e.*, an area of concrete equal to the product of the steel area and the ratio E_s/E_c which may generally be taken as 15. The resulting section to be dealt with is similar to that shown in Fig. 8, the fins being opposite the steel bars. If the resultant at any section of a reinforced-concrete arch passes outside the kernel, special formulas must be applied, for which see works upon reinforced concrete mentioned in references at end of Chap. II.

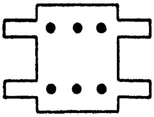


FIG. 8.

84. Illustrative Example. The following example illustrates the method of application of the equations of this chapter to a 1-ft. slice of the arch shown in Fig. 9 subjected to a uniform load of 200 lb. per linear foot applied to the left half of the span. Section numbers refer to sections on left half of span.

TABLE I

Section	z in feet, scaled	d in feet, scaled	d^3	$1/d^3$	z/d^3	$1/d$
1	20.49	1.70	4.91	0.204	4.18	0.588
2	20.18	1.70	4.91	0.204	4.12	0.588
3	19.45	1.70	4.91	0.204	3.97	0.588
4	18.45	1.74	5.27	0.190	3.51	0.575
5	17.08	1.75	5.36	0.187	3.19	0.571
6	15.37	1.78	5.64	0.177	2.72	0.562
7	13.38	2.00	8.00	0.125	1.67	0.500
8	10.84	2.45	14.71	0.068	0.74	0.408
9	7.98	3.05	28.37	0.035	0.28	0.328
10	4.70	4.00	64.00	0.016	0.07	0.250
Total.....	.....	....	.....	1.410	24.45	4.958

$$t = \frac{24.45}{1.41} = 17.34 \text{ ft.} \qquad h - t = 20.5 - 17.34 = 3.16 \text{ ft.}$$

In the solution the arch was laid out to a large scale and the constants z, d, x, and y given in Tables I and II scaled from the

TABLE II

Section	x Scaled	$y = z - t$	x^2	y^2	x/d^3	y/d^3	x^2/d^3	y^2/d^3
1	2.84	+ 3.15	8.0	9.9	0.58	+0.642	1.6	2.02
2	8.48	+ 2.84	71.9	8.1	1.73	+0.579	14.7	1.64
3	14.10	+ 2.11	198.8	4.4	2.88	+0.430	40.5	0.91
4	19.68	+ 1.11	387.3	1.2	3.74	+0.211	73.5	0.23
5	25.18	− 0.26	634.0	0.1	4.70	−0.048	118.2	0.01
6	30.55	− 1.97	933.3	3.9	5.41	−0.349	165.5	0.69
7	35.80	− 3.96	1281.6	15.7	4.48	−0.495	160.2	1.96
8	40.89	− 6.50	1672.0	42.2	2.78	−0.442	113.8	2.87
9	45.69	− 9.36	2087.6	87.6	1.61	−0.329	73.6	3.09
10	50.30	−12.64	2530.1	159.8	0.79	−0.198	39.5	2.50
Total......	.....	....	......	.....	....	....	801.1	15.92

TABLE III

Live load over left half of arch; moments in 1000-lb. units

Section	m_L	m_R	$(m_L + m_R)$	$(m_L - m_R)$	$(m_L + m_R)\frac{1}{d^3}$	$(m_L - m_R)\frac{x}{d^3}$	$(m_L + m_R)\frac{y}{d^3}$
1	0.8	0	0.8	0.8	0.16	0.5	+ 0.5
2	7.2	0	7.2	7.2	1.46	12.4	+ 4.2
3	19.9	0	19.9	19.9	4.05	57.2	+ 8.5
4	38.7	0	38.7	38.7	7.35	144.9	+ 8.2
5	63.4	0	63.4	63.4	11.81	297.8	− 3.0
6	93.3	0	93.3	93.3	16.52	504.0	− 32.6
7	128.2	0	128.2	128.2	16.05	575.0	− 63.5
8	167.2	0	167.2	167.2	11.37	465.0	− 73.9
9	208.8	0	208.8	208.8	7.36	336.0	− 68.6
10	253.0	0	253.0	253.0	3.95	198.5	− 50.2
Total..	.....	..	.....	.....	80.08	2591.3	−270.4

$$M_0 = \frac{1}{2}\,\frac{80{,}080}{1.41} = 28{,}400 \text{ ft.-lb.}$$

$$H_0 = \frac{1}{2}\,\frac{270{,}400}{15.92 + 0.41} = 8270 \text{ lb.}$$

$$V_0 = \frac{1}{2}\,\frac{2{,}591{,}300}{801.1} = 1615 \text{ lb.}$$

$$M_c = 28{,}400 - 8270 \times 3.16 = +2270 \text{ ft.-lb.}$$

layout. The values of m_L at the different sections were next found by applying the equation $m_L = wx^2/2$ and are shown in Table III. The values of m_R are 0 in all cases owing to the fact that there is no load on the right half of the structure. The terms m_L and m_R are the only values affected by the loading.

From the values given in Table III combined with those in Tables I and II the values of M_0, H_0, V_0, and M_c were easily determined.

Had the problem been that of determining the effect of the dead loads the magnitude of these loads and their points of application should have been determined by the methods outlined in Art. 75.

Had the arch been loaded through columns instead of by an earth fill the points of application of the floor loads and the live loads should have been taken at the centre lines of the columns.

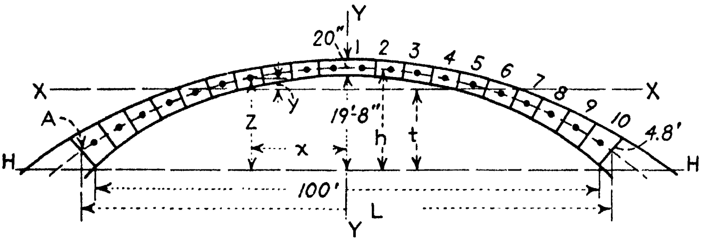


FIG. 9.

85. Influence Lines.—Influence lines for moments or thrusts at any section of a fixed-ended arch may be readily constructed, and the position of loads for maximum value of the function in question easily determined. With the position of the loads known, the value of the function itself can be determined for a uniform loading, by multiplying the area between the influence line and the horizontal axis by the uniform load per foot. For a concentrated load system the value of the function can be obtained by placing the loads on the span and getting the combined product of loads and ordinates to the influence line. The determination of the ordinate to the influence line for a typical case is illustrated by the following example:

Problem: Determine the ordinate to the influence line for moment at section 5 (practically the quarter point) of the left half of the arch shown in the previous example with the load at the section.

Solution: For such cases tables similar to Table III may be constructed. m_R will be zero in all cases, and m_L will also be zero for sections 1 to 5, inclusive, since in all of foregoing cases the load will be to right of or at the section under consideration. The remaining values are shown in Table IV which follows:

TABLE IV.—LOAD AT SECTION 5

Section	m_L	$(m_L + m_R)\frac{1}{d^3}$	$(m_L + m_R)\frac{x}{d^3}$	$(m_L + m_R)\frac{y}{d^3}$
6	5.37	0.95	29.1	− 1.87
7	10.62	1.33	47.6	− 5.26
8	15.71	1.07	43.7	− 6.94
9	20.51	0.72	33.0	− 6.75
10	25.12	0.40	19.8	− 4.97
Total....	.....	4.47	173.2	−25.79

By means of the figures in the various tables the values of M_0, H_0, and V_0 are found by inserting the tabular values in Eqs. (10*a*), (11*a*) and (12*a*), and are as follows:

$$M_0 = \frac{1}{2}\left(\frac{4.47}{1.41}\right) = +1.58$$

$$H_0 = -\frac{1}{2}\left(-\frac{25.79}{15.92 + 0.4}\right) = +0.79$$

$$V_0 = \frac{1}{2}\left(\frac{173.2}{801.1}\right) = +0.11$$

The moment at the section under consideration which equals the ordinate to the influence line at this section =

$$M_0 + H_0(17.34 - 17.08) + V_0(25.18) = 4.51$$

The ordinate to the influence line for moment at the point under consideration due to load unity at other points on the arch may be computed in a similar manner, the influence line constructed, and the areas between influence line and axis determined by planimeter or otherwise.

Figure 10 shows the complete influence line for moment at section 5 on left half of arch and also for moment at left end and at crown and the areas between these various influence lines and the axis.

For uniform live load equal w per foot the maximum moment at section 5 occurs with load over distance FB and equals $-66.7w$ ft.-lb.; the maximum moment at A occurs with loading from C to F and equals $+336.0w$ ft. lb.; and the maximum crown moment occurs with loading over distance equal to DE on each side of crown and equals $55.5w$ ft.-lb.

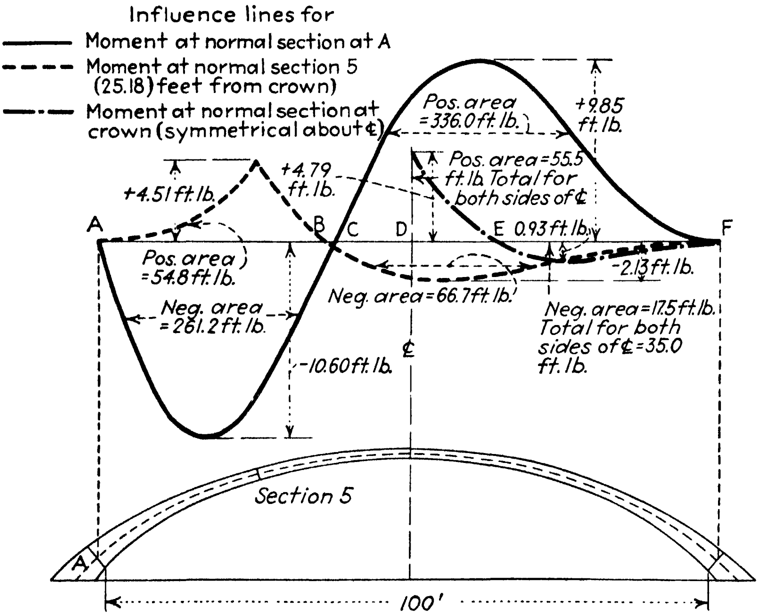

Influence lines for
Moment at normal section at A
Moment at normal section 5 (25.18) feet from crown)
Moment at normal section at crown (symmetrical about ℄)
Pos. area =336.0 ft. lb.
+9.85 ft. lb.
+4.79 ft. lb.
+4.51 ft. lb.
Pos. area =55.5 ft. lb. Total for both sides of ℄
A
B
C
D
E
F
0.93 ft. lb.
Pos. area =54.8 ft. lb.
-2.13 ft. lb.
Neg. area =66.7 ft. lb.
Neg. area =261.2 ft. lb.
Neg. area =17.5 ft. lb. Total for both sides of ℄ =35.0 ft. lb.
℄
-10.60 ft. lb.
Section 5
A
100′

FIG. 10.

86. Critical Section.—Since, for appearance, the arch rib is generally made of a constantly increasing thickness from the crown to the skewback where the moment is a maximum, it is generally sufficient to compute the maximum moments, thrusts, and resulting stresses at crown and skewback. The moment at the quarter point is seldom much larger than the moment at the crown. Influence lines for these three points for various arches are given in the papers by Whitney and Cochrane referred to in the references at the end of the chapter.

87. Deflection and Change of Slope of Symmetrical Fixed-ended Arch.—The linear deflection or change of slope of any point on the arch axis may be found by applying the general equations given in Art. 79, *viz.:*

$$\text{Deflection} = \int \frac{Mm_0 ds}{EI}$$

$$\text{Change of slope} = \int \frac{Mm_1 ds}{EI}$$

The arch should be considered as cut at the crown, and the free end of each half connected by rigid arms to points b, as shown in Figs. 6 and 7, at which points the forces M_0, H_0, and V_0 are acting. Of course, before the deflection of a point may be found, the values of M_0, H_0, and V_0 must first be evaluated for the load condition, temperature change and abutment movement being considered. Then M, m_0, and m_1 are all functions of known quantities so that the foregoing integrals may be evaluated to obtain the desired deflection.

If the desired deflection or change of slope is to be obtained under a condition that involves an abutment movement or temperature change, the deflection or change of slope due to the abutment movement or temperature change must be added algebraically to the values obtained from the foregoing integrals.

It should also be noted that the preceding discussion has not considered the effect of axial thrust upon the deflection or change of slope. If it is desired to include this effect, which is small, it may be done in a manner similar to that used in Chap. VII.

The details of this subject will be left to the reader.

88. Unsymmetrical Fixed-ended Arch. General.—One type of unsymmetrical fixed-ended arch has already been discussed in Art. 79. However, for the general case of an unsymmetrical fixed-ended arch, the same method of attack may be used as in the case of the symmetrical arch except that a system of coordinate axes must be selected such that their origin is at some point b in Fig. 6 and 7 but so located that $\int \frac{xds}{EI} = 0$ and $\int \frac{yds}{EI} = 0$ and that the y axis is vertical and the x axis sloping such that $\int \frac{xyds}{EI} = 0$. The x coordinate should be measured perpendicular to the y axis, and the y coordinate perpendicular to the x axis. Then the three unknowns will act at point b', V_0 along the y axis, H_0 along the x axis, and M_0 as before.

Also, it will be no longer correct to take the coordinate x *as positive both to left and to right* but will be more convenient to adopt one direction as positive for the x coordinate. It is also evident that it will be no longer possible to take advantage of symmetry in evaluating the integrals which will occur in the expressions for M_0, H_0, and V_0.

The reader will probably find that the advantages of working from a point b located as previously indicated for an unsymmetrical arch will be outweighed by the complexities involved. It will perhaps be simpler to select the three unknowns either as the moment, horizontal, and vertical reactions at one abutment or the moment, shear, and thrust at the section of the arch where the tangent to the axis is horizontal, *i.e.*, the crown of the arch, on the assumption that this point will lie within the span. The derivation of the equations for the three unknowns selected in either of these ways will be more easily understood and retained by the reader.

The details of deriving equations for the three unknowns selected will be left to the reader, with the suggestion that he refer to references given at the end of Chap. II for assistance.

89. Continuous Arches on Elastic Supports.—Arch bridges continuous over several spans and supported on elastic piers and abutments may be readily solved by the method of least work.

General equations for such a structure were developed in an unpublished paper by K. B. Skardon, S.M., M.I.T., 1925, and are given in this article; their derivation appears in Appendix E. Equations are given both for the case of vertical loads and for temperature change, the effect of the latter being especially important. In developing these equations, the piers and abutments have been considered as fixed against translation and rotation at their bases and as uniform in thickness. If battered piers and abutments are used, approximate integration or summation should be employed in developing the equations, as has been done in the case of the fixed-ended arch ribs.

For structures of this type, the degree of indeterminacy is equal to three times the number of spans. For each span, redundants may be taken as the moment, shear, and thrust at the crown, and three equations may be developed. These equations may be applied to any span by changing the subscripts, as is done with the three-moment equation; hence the number of equations available for the solution of any such structure equals the number of unknowns.

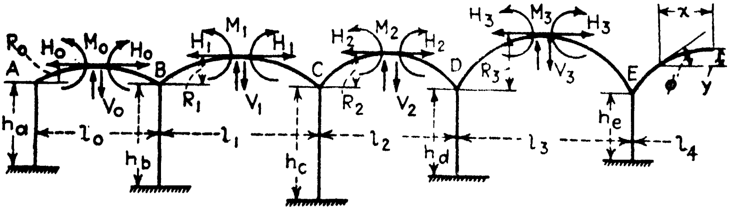


FIG. 11.—Each arch symmetrical.

Equations (25), (26), and (27) apply to vertical loads only, and Eqs. (28), (29), and (30) to temperature or shrinkage effects.

Origin of coordinates for each arch is taken at the crown. All units are in pounds, feet, and foot-pounds.

Direct stress in arch rib is assumed as constant and equal to H_0, etc., throughout the length of arch.[1]

Let x and y = coordinates of any point on the arch axis referred to the origin; x positive to the left or to the right, and y positive downward.

s_0, s_1, etc. = lengths of each arch axis.

M_0, H_0, V_0, M_1, H_1, V_1, etc. = crown forces for each arch, positive when acting as shown on Fig. 11.

R_0, etc. = rise of each arch.

W_A = sum of all vertical loads applied to arch l_0 between A and the crown.

W_B = sum of all vertical loads applied to arches l_0 and l_1 between their respective crowns.

W_C, etc., be similar to W_B but applied to other spans.

M_A = moment of all vertical loads between A and crown of arch l_0 taken about point A, half arch being assumed to act as a cantilever. Positive when clockwise.

M_B = algebraic moment of all vertical loads between crown of l_0 and crown of l_1 taken about B, each half arch being assumed to act as a cantilever. Positive when clockwise.

---

[1] **This is only slightly approximate. See Theory of Structures, 3d ed. Spofford, p. 534.**

M_C, etc., be similar to M_B but applied to other spans.

m_{L_0} = moment at any point x, y in the left half of arch l_0 of all loads between x, y and the crown, left half of arch being assumed to be a cantilever beam.

m_{R_0} be similar to m_{L_0} for right side of arch l_0.

m_{L_1}, m_{L_2}, etc., be similar to m_{L_0} but applied to other spans.

m_{R_1}, m_{R_2}, etc., be similar to m_{R_0} but applied to other spans.

I_0, I_1, etc. = moments of inertia of normal cross sections of the arches l_0, l_1, etc., at any point x, y.

I_a, I_b, etc. = respective moments of inertia of the piers and abutments, which for the purpose of this demonstration are taken as constant.

A_0, A_1, etc. = areas of normal cross sections of the arches l_0, l_1, etc., at any point x, y.

A_a, A_b, etc. = areas of the piers and abutments.

E = modulus of elasticity of the material in the arch ribs and piers.

Δ_s = length of sections into which arches, piers, and abutments are assumed to be divided.

ϵ = coefficient of expansion of material.

t = temperature change in degrees Fahrenheit, positive when an increase.

Using the method of least work, the following three general equations are obtained for the case of vertical loads by taking the partial derivative of the total internal work in the structure with respect to M_1, H_1, and V_1, respectively for the case where E is constant:

$$\begin{aligned}
0 = &+M_0\left(-\frac{h_b}{I_b}\right) + H_0\left(-\frac{R_0h_b}{I_b} - \frac{h_b^2}{2I_b}\right) + V_0\left(\frac{l_0h_b}{2I_b}\right) \\
&+ M_1\left(2\sum_0^{\frac{S_1}{2}}\frac{\Delta s_1}{I_1} + \frac{h_b}{I_b} + \frac{h_c}{I_c}\right) \\
&+ H_1\left(2\sum_0^{\frac{S_1}{2}}\frac{y\Delta s_1}{I_1} + \frac{R_1h_b}{I_b} + \frac{h_b^2}{2I_b} + \frac{R_1h_c}{I_c} + \frac{h_c^2}{2I_c}\right) \\
&+ V_1\left(\frac{l_1h_b}{2I_b} - \frac{l_1h_c}{2I_c}\right) + M_2\left(-\frac{h_c}{I_c}\right) + H_2\left(-\frac{R_2h_c}{I_c} - \frac{h_c^2}{2I_c}\right) \\
&+ V_2\left(-\frac{l_2h_c}{2I_c}\right) - \sum_0^{\frac{S_1}{2}}(m_{L_1} + m_{R_1})\frac{\Delta s_1}{I_1} - \frac{M_Bh_b}{I_b} + \frac{M_Ch_c}{I_c} \qquad (25)
\end{aligned}$$

$$
\begin{aligned}
0 = {} & M_0\left(-\frac{R_1h_b}{I_b} - \frac{h_b^2}{2I_b}\right) + H_0\left(-\frac{R_0R_1h_b}{I_b} - \frac{R_1h_b^2}{2I_b} - \frac{R_0h_b^2}{2I_b} - \frac{h_b^3}{3I_b}\right) \\
& + V_0\left(\frac{R_1l_0h_b}{2I_b} + \frac{l_0h_b^2}{4I_b}\right) + \\
& \qquad M_1\left(2\sum_0^{\frac{S_1}{2}}\frac{y\Delta s_1}{I_1} + \frac{R_1h_b}{I_b} + \frac{h_b^2}{2I_b} + \frac{R_1h_c}{I_c} + \frac{h_c^2}{2I_c}\right) \\
& + H_1\left(2\sum_0^{\frac{S_1}{2}}\frac{y^2\Delta s_1}{I_1} + 2\sum_0^{\frac{S_1}{2}}\frac{\Delta s_1}{A_1} + \frac{R_1^2h_b}{I_b} + \frac{R_1h_b^2}{I_b} + \right. \\
& \qquad \left. \frac{h_b^3}{3I_b} + \frac{R_1^2h_c}{I_c} + \frac{R_1h_c^2}{I_c} + \frac{h_c^3}{3I_c}\right) \\
& + V_1\left(\frac{R_1l_1h_b}{2I_b} + \frac{l_1h_b^2}{4I_b} - \frac{R_1l_1h_c}{2I_c} - \frac{l_1h_c^2}{4I_c}\right) + M_2\left(-\frac{R_1h_c}{I_c} - \frac{h_c^2}{2I_c}\right) \\
& + H_2\left(-\frac{R_1R_2h_c}{I_c} - \frac{R_2h_c^2}{2I_c} - \frac{R_1h_c^2}{2I_c} - \frac{h_c^3}{3I_c}\right) + \\
& \qquad V_2\left(-\frac{l_2R_1h_c}{2I_c} - \frac{l_2h_c^2}{4I_c}\right) \\
& - \sum_0^{\frac{S_1}{2}}(m_{L_1} + m_{R_1})\frac{y\Delta s_1}{I_1} - \frac{M_BR_1h_b}{I_b} \quad \frac{M_Bh_b^2}{2I_b} + \\
& \qquad \frac{M_CR_1h_c}{I_c} + \frac{M_Ch_c^2}{2I_c} \qquad (26)
\end{aligned}
$$

$$
\begin{aligned}
0 = {} & M_0\left(-\frac{l_1h_b}{2I_b}\right) + H_0\left(-\frac{R_0l_1h_b}{2I_b} - \frac{l_1h_b^2}{4I_b}\right) + V_0\left(\frac{l_0l_1h_b}{4I_b} - \frac{h_b}{A_b}\right) \\
& + M_1\left(\frac{l_1h_b}{2I_b} - \frac{l_1h_c}{2I_c}\right) + H_1\left(\frac{R_1l_1h_b}{2I_b} + \frac{l_1h_b^2}{4I_b} - \frac{R_1l_1h_c}{2I_c} - \frac{l_1h_c^2}{4I_c}\right) \\
& + V_1\left(2\sum_0^{\frac{S_1}{2}}\frac{x^2\Delta s_1}{I_1} + \frac{l_1^2h_b}{4I_b} + \frac{l_1^2h_c}{4I_c} + \frac{h_b}{A_b} + \frac{h_c}{A_c}\right) + M_2\left(\frac{l_1h_c}{2I_c}\right) \\
& + H_2\left(\frac{R_2l_1h_c}{2I_c} + \frac{l_1h_c^2}{4I_c}\right) + V_2\left(\frac{l_1l_2h_c}{4I_c} - \frac{h_c}{A_c}\right) \\
& - \sum_0^{\frac{S_1}{2}}(m_{L_1} - m_{R_1})\frac{x\Delta s_1}{I_1} - \frac{M_Bl_1h_b}{2I_b} - \frac{W_Bh_b}{A_b} - \\
& \qquad \frac{M_Cl_1h_c}{2I_c} + \frac{W_Ch_c}{A_c} \qquad (27)
\end{aligned}
$$

Three equations of this type may be written for each span by simply changing the subscripts. Thus sufficient equations are obtained which when solved simultaneously give the values of the redundants.

For the case of a uniform change in temperature over the entire structure, three equations of the same type as the preceding are obtained as follows:

$$\begin{aligned}0 = {} & M_0\left(-\frac{h_b}{I_b}\right) + H_0\left(-\frac{R_0h_b}{I_b} - \frac{h_b^2}{2I_b}\right) + V_0\left(\frac{l_0h_b}{2I_b}\right) + \\ & M_1\left(2\sum_0^{\frac{s_1}{2}}\frac{\Delta s_1}{I_1} + \frac{h_b}{I_b} + \frac{h_c}{I_c}\right) \\ & + H_1\left(2\sum_0^{\frac{s_1}{2}}\frac{y\Delta s_1}{I_1} + \frac{R_1h_b}{I_b} + \frac{h_b^2}{2I_b} + \frac{R_1h_c}{I_c} + \frac{h_c^2}{2I_c}\right) + \\ & V_1\left(\frac{l_1h_b}{2I_b} - \frac{l_1h_c}{2I_c}\right) \\ & + M_2\left(-\frac{h_c}{I_c}\right) + H_2\left(-\frac{R_2h_c}{I_c} - \frac{h_c^2}{2I_c}\right) + V_2\left(-\frac{l_2h_c}{2I_c}\right)\end{aligned} \tag{28}$$

$$\begin{aligned}0 = {} & M_0\left(-\frac{R_1h_b}{I_b} - \frac{h_b^2}{2I_b}\right) + H_0\left(-\frac{R_0R_1h_b}{I_b} - \frac{R_1h_b^2}{2I_b} - \frac{R_0h_b^2}{2I_b} - \frac{h_b^3}{3I_b}\right) \\ & + V_0\left(\frac{l_0R_1h_b}{2I_b} + \frac{l_0h_b^2}{4I_b}\right) + M_1\left(2\sum_0^{\frac{s_1}{2}}\frac{y\Delta s_1}{I_1} + \frac{R_1h_b}{I_b} + \right. \\ & \left. \frac{h_b^2}{2I_b} + \frac{R_1h_c}{I_c} + \frac{h_c^2}{2I_c}\right) \\ & + H_1\left(2\sum_0^{\frac{s_1}{2}}\frac{y^2\Delta s_1}{I_1} + 2\sum_0^{\frac{s_1}{2}}\frac{\Delta s_1}{A_1} + \frac{R_1^2h_b}{I_b} + \frac{R_1h_b^2}{I_b} + \frac{h_b^3}{3I_b} + \right. \\ & \left. \frac{R_1^2h_c}{I_c} + \frac{R_1h_c^2}{I_c} + \frac{h_c^3}{3I_c}\right) \\ & + V_1\left(\frac{l_1R_1h_b}{2I_b} + \frac{l_1h_b^2}{4I_b} - \frac{l_1R_1h_c}{2I_c} - \frac{l_1h_c^2}{4I_c}\right) \\ & + M_2\left(-\frac{R_1h_c}{I_c} - \frac{h_c^2}{2I_c}\right) + \\ & H_2\left(-\frac{R_1R_2h_c}{I_c} - \frac{R_1h_c^2}{2I_c} - \frac{R_2h_c^2}{2I_c} - \frac{h_c^3}{3I_c}\right) \\ & + V_2\left(-\frac{l_2R_1h_c}{2I_c} - \frac{l_2h_c^2}{4I_c}\right) - etl_1E\end{aligned} \tag{29}$$

$$0 = M_0\left(-\frac{l_1h_b}{2I_b}\right) + H_0\left(-\frac{R_0l_1h_b}{2I_b} - \frac{l_1h_b^2}{4I_b}\right) + V_0\left(\frac{l_0l_1h_b}{4I_b} - \frac{h_b}{A_b}\right)$$

$$
\begin{aligned}
&+ M_1\left(\frac{l_1h_b}{2I_b} - \frac{l_1h_c}{2I_c}\right) + H_1\left(\frac{R_1l_1h_b}{2I_b} + \frac{l_1h_b^2}{4I_b} - \frac{R_1l_1h_c}{2I_c} - \frac{l_1h_c^2}{4I_c}\right) \\
&+ V_1\left(2\sum_0^{\frac{s_1}{2}} \frac{x^2\Delta s_1}{I_1} + \frac{l_1^2h_b}{4I_b} + \frac{h_b}{A_b} + \frac{l_1^2h_c}{4I_c} + \frac{h_c}{A_c}\right) + M_2\left(\frac{l_1h_c}{2I_c}\right) \\
&+ H_2\left(\frac{R_2l_1h_c}{2I_c} + \frac{l_1h_c^2}{4I_c}\right) + V_2\left(\frac{l_2l_1h_c}{4I_c} - \frac{h_c}{A_c}\right) \\
&+ \epsilon tE(h_b - h_c) \qquad (30)
\end{aligned}
$$

These last three equations are used in the same manner as the first three, the results giving the values of the redundants due to a change in temperature. (See Art. 44 for effect of shrinkage.)

90. Problem. Three-span Continuous Arch on Elastic Supports.—The following example from Skardon's paper illustrates the application of Eqs. (25), (26), and (27) to the solution of the dead stresses in a reinforced-concrete bridge of three flat-arch spans on elastic piers and abutments fixed at the foundations. The structure is symmetrical about the centre line of the centre span and is shown in half-longitudinal section by Fig. 12. In all calculations a longitudinal strip 1 ft. wide is considered.

Solution: Each half of each arch axis is divided into 10 equal parts Δs_0, Δs_1, etc., for the purpose of performing the indicated approximate summation. Measurement for these calculations were scaled from the drawing (Fig. 12) and are given in Table V.

Moments of inertia are determined using the entire plain concrete sections which give a close approximation to the transformed steel and concrete sections not yet designed.

Applying Eq. (25) to span l_0 by changing subscripts and omitting terms that apply to a span to the left of l_0, which does not exist, gives Eq. (1) of Table VI, *viz.:*

$$
\begin{aligned}
0 &= M_0\left(2 \times 3.8 \times 6.477 + \frac{20.8}{10.4} + \frac{24.1}{10.4}\right) = && +53.52M_0 \\
&+ H_0\left(2 \times 3.8 \times 1.927 + \frac{2.5 \times 20.8}{10.4} + \frac{\overline{20.8}^2}{2 \times 10.4} + \frac{2.5 \times 24.1}{10.4} + \frac{\overline{24.1}^2}{2 \times 10.4}\right) = && +74.17H_0 \\
&+ V_0\left(\frac{76.25 \times 20.8}{2 \times 10.4} - \frac{76.25 \times 24.1}{2 \times 10.4}\right) = && -12.10V_0 \\
&+ M_1\left(-\frac{24.1}{10.4}\right) = && -2.32M_1 \\
&+ H_1\left(-\frac{3.0 \times 24.1}{10.4} - \frac{\overline{24.1}^2}{2 \times 10.4}\right) = && -34.86H_1 \\
&+ V_1\left(-\frac{79.25 \times 24.1}{2 \times 10.4}\right) = && -91.8V_1 \\
&- 3.8 \times 376{,}700 - \frac{337{,}500 \times 20.8}{10.4} + \frac{42{,}800 \times 24.1}{10.4} = && -2{,}007{,}000
\end{aligned}
$$

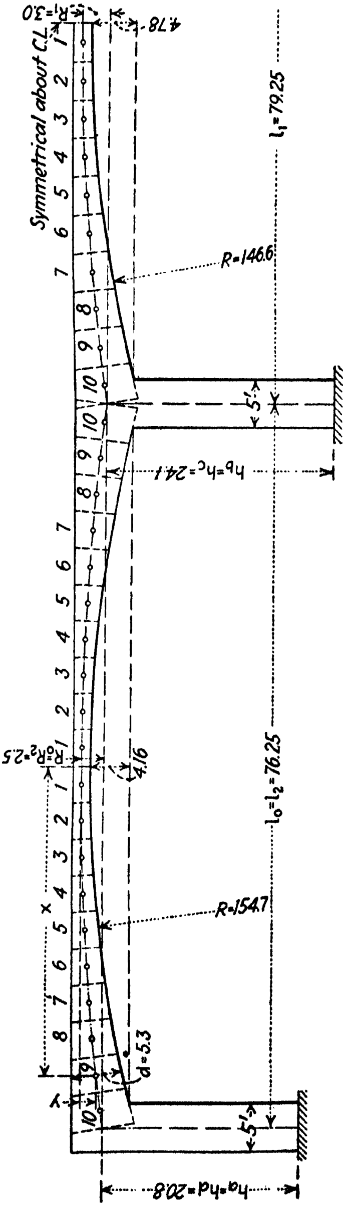

Symmetrical about C.L.
R₁=3.0
4.78
l₁=79.25
R=146.6
5'
h_b=h_c=24.1
4.16
R₀=R₂=2.5
l₀=l₂=76.25
R=154.7
x
y
d=5.3
5'
h_a=h_d=20.8
1
2
3
4
5
6
7
8
9
10
10
9
8
7
6
5
4
3
2
1
1
2
3
4
5
6
7
8
9
10

FIG. 12.

TABLE. V.—DATA FOR DEAD-LOAD STRESSES

Spans l_0 and l_2

Point	d	d^3	$\frac{12}{d^3}$	$\frac{1}{d}$	x	x^2	$\frac{12x^2}{d^3}$	y	y^2	$\frac{12y}{d^3}$	$\frac{12y^2}{d^3}$	$\frac{12x}{d^3}$	$m_L = m_R$	$m_L + m_R$	$(m_L + m_R)\frac{12}{d^3}$	$(m_L + m_R)\frac{12y}{d^3}$
1	1.9	6.86	1.749	0.526	1.9	3.6	6.3	0.0	0.00	0.000	0.000	3.32	630	1.260	2,200	0
2	2.0	8.00	1.500	0.500	5.7	32.5	48.8	0.1	0.01	0.150	0.015	8.55	5,710	11,420	17,150	1,710
3	2.2	10.65	1.127	0.455	9.5	90.3	101.9	0.2	0.04	0.225	0.045	10.71	16,090	32,180	36,200	7,240
4	2.5	15.63	0.767	0.400	13.4	179.6	137.9	0.4	0.16	0.307	0.123	10.28	32,200	64,400	49,400	19,800
5	2.8	21.95	0.546	0.357	17.2	295.8	161.4	0.6	0.36	0.328	0.197	9.39	54,630	109,260	59,700	35,800
6	3.3	35.94	0.334	0.303	21.0	411.0	147.3	0.8	0.64	0.267	0.214	7.01	84,140	168.280	56,200	44,900
7	3.9	59.32	0.202	0.256	24.8	615.0	124.3	1.1	1.21	0.222	0.244	5.01	121,760	243,520	49,100	54,000
8	4.6	97.34	0.123	0.217	28.7	823.7	101.3	1.4	1.96	0.172	0.241	3.53	168,780	337,560	41,500	58,000
9	5.3	148.88	0.081	0.189	32.5	1,056.3	85.5	1.8	3.24	0.146	0.263	2.63	226,700	453,400	36,750	66.100
10	6.3	250.05	0.048	0.159	36.3	1,317.7	63 2	2.3	5.29	0.110	0.253	1.74	297,140	594.280	28,500	65,300
			6.477	3.362			977.9			1.927	1.595				376,700	352,850
Span l_1																
1	1.9	6.86	1.749	0.526	2.0	4.0	7.0	0.0	0.00	0.000	0.000	3.50	700	1,400	2,450	0
2	2.1	9.26	1.294	0.476	6.0	36.0	46.6	0.1	0.01	0.129	0.013	7.76	6,380	12,760	16.500	1,650
3	2.3	12.17	0.985	0.435	10.0	100.0	98.5	0.3	0.09	0.295	0.089	9.85	18,120	36,240	35,800	10,700
4	2.6	17.58	0.683	0.385	14.0	196.0	134.0	0.5	0.25	0.342	0.171	9.56	36,440	72,880	49,800	24,900
5	2.9	24.39	0.492	0.345	18.0	324.0	159.4	0.7	0.49	0.344	0.241	8.86	62,020	124,040	61,100	42,700
6	3.4	39.30	0.305	0.294	21.9	479.6	146.2	0.9	0.81	0.275	0.247	6.68	95,680	191,360	58,400	52,600
7	4.1	68.92	0.174	0.244	25.9	670.8	116.9	1.3	1.69	0.226	0.294	4.51	138,620	277,240	48,200	62,600
8	4.7	103.82	0.116	0.213	29.9	894.0	103.9	1.7	2.89	0.197	0.335	3.47	192,360	384,720	44,600	75,800
9	5.5	166.38	0.072	0.182	33.9	1,149.2	82.6	2.2	4.84	0.158	0.348	2.44	258,500	517,000	37,200	81,600
10	6.4	262.14	0.046	0.156	37.8	1,428.8	65.7	2.7	7.29	0.124	0.335	1.74	338,920	677.840	31,200	84,000
			5.916	3.256			960.8			2.090	2.073				385,250	436,550

$h_a = h_d = 20.8$
$h_b = h_c = 24.1$
$R_0 = R_2 = 2.5$
$R_1 = 3.0$

$l_0 = 76.25$
$l_1 = 79.25$
$l_2 = 76.25$
$\Delta s_0 = \Delta s_2 = 3.8$
$\Delta s_1 = 4.0$

$I_a, I_b, I_c, I_d = 10.4$
$A_a, A_b, A_c, A_d = 5.0$
$M_A = +337{,}500$ ft.-lb.
$M_B = +42{,}800$ ft.-lb.
$M_C = -42{,}800$ ft.-lb.

$M_D = 337{,}500$ ft.-lb.
$W_A = 22{,}120$ lb.
$W_B = 46{,}120$ lb.
$W_C = 46{,}120$ lb.
$W_D = 22{,}120$ lb.

TABLE VI.—SIMULTANEOUS SOLUTION OF DEAD-LOAD EQUATION

Equation	M_0	+ H_0	+ V_0	+ M_1	+ H_1	+ V_1	+ M_2	+ H_2	+ V_2	= C (00 omitted)
1	53.52	74.17	-12.10	-2.32	-34.86	-91.8	....	....	....	2,007.0
2	74.17	1,036.0	-301.8	-33.71	-619.57	-1,336.0	....	....	....	8,605.0
3	-12.10	-301.8	13,716.0	88.40	1,330.0	3,495.0	....	....	....	29,381.0
4	-2.32	-33.71	88.40	51.95	86.44	0	-2.32	-33.71	-88.40	1,739.0
5	-34.86	-619.57	1,330.0	86.44	1,317.0	0	-34.86	-619.57	-1,330.0	4,730.0
6	-91.80	-1,336.0	3,495.0	0	0	14,975.0	91.80	1,336.0	3,495.0	0.0
7	....	....	....	-2.32	-34.86	91.8	53.52	74.17	12.10	2,005.0
8	....	....	....	-33.71	-619.57	1,336.0	74.17	1,045.0	302.3	8,605.0
9	....	....	....	-88.4	-1,330.0	3,495.0	12.10	302.3	13,710.0	-29,381.0
1	53.52	74.17	-12.10	-2.32	-34.86	-91.80	....	....	....	2,007.0
2 × 0.7215855	53.52	747.56	-217.77	-24.32	-447.07	-964.04	....	....	....	6,209.2
1 - 2 = Eq. 10	....	-673.39	205.67	22.00	412.21	872.24	....	....	....	-4,202.2
2	74.17	1,036.00	-301.8	-33.71	-619.57	-1,336.0	....	....	....	8,605.0
3 × 6.129752	-74.17	-1,850.0	84,075.7	541.87	8,152.57	21,423.5	....	....	....	180,098.2
2 + 3 = Eq. 11	....	-814.0	83,773.9	508.16	7,533.00	20,087.5	....	....	....	188,703.2
3	-12.10	-301.8	13,716.0	88.40	1,330.0	3,495.0	....	....	....	29,381.0
4 × 5.215517	-12.10	-175.8	461.1	270.95	450.8	0	-12.10	-175.8	-461.1	9,069.8
3 - 4 = Eq. 12	....	-126.0	13,254.9	-182.55	879.2	3,495.0	12.10	175.8	461.1	20,311.2
5	-34.86	-619.57	1,330.0	86.44	1,317.0	0	-34.86	-619.57	-1,330.0	4,730.0
3 × 2.8809917	-34.86	-869.48	39,515.7	254.68	3,831.7	10,069.0	....	....	....	84,646.4
3 - 5 = Eq. 13	....	-249.91	38,185.7	168.24	2,514.7	10,069.0	34.86	619.57	1,330.0	79,916.4
6	-91.80	-1,336.0	3,495.0	0	0	14,975.0	91.80	1,336.0	3,495.0	0
1 × 1.7152466	91.80	127.2	-20.8	-3.98	-59.79	-157.5	....	....	....	3,442.5
6 + 1 = Eq. 14	....	-1,208.8	3,474.2	-3.98	-59.79	14,817.5	91.80	1,336.0	3,495.0	3,442.5
10	....	-673.39	205.67	22.00	412.21	872.24	....	....	....	-4,202.2
11	....	-814.0	83,773.9	508.16	7,533.0	20,087.5	....	....	....	188,703.2
12	....	-126.0	13,254.9	-182.55	879.2	3,495.0	12.10	175.8	461.1	20,311.2
13	....	-249.91	38,185.7	168.24	2,514.7	10,069.0	34.86	619.57	1,330.0	79,916.4
14	....	-1,208.8	3,474.2	-3.98	-59.79	14,817.5	91.80	1,336.0	3,495.0	3,442.5

By similar substitutions, Eqs. (26) and (27) when applied to span l_0 reduce to the form shown in Table VI. Equations (25) to (27) when applied similarly to span l_1 and l_2 complete the nine independent equations required to solve the dead-load stresses. In Table VI the constant terms are placed in the right-hand column. A sufficient amount of the computations are shown to indicate the method used for solution. Since all succeeding groups of simultaneous equations are solved in a similar manner, the actual computations are omitted beyond this point, and only results are recorded.

Equations (10) to (14) may now be combined to eliminate H_0 and the process continued until all but V_2 are eliminated, giving the value of V_2. Tabular Eqs. (7), (8), and (9) must also be picked up and included in the solution as soon as M_0, H_0, and V_0 are eliminated. The calculations should be made on a machine to give the accuracy required in the final results. After V_2 is determined, its value may be substituted back into the last group of equations to find H_2. The values of V_2 and H_2 may then be substituted into the next group of equations, and M_2 found. This process may be continued until all unknowns are determined, giving the following values in pounds and foot-pounds respectively:

$$\begin{array}{lll} V_2 = -320 & V_1 = 0 & V_0 = +320 \\ H_2 = +20{,}620 & H_1 = +22{,}010 & H_0 = +20{,}620 \\ M_2 = +24{,}410 & M_1 = +24{,}690 & M_0 = +24{,}410 \end{array}$$

The structure under consideration is symmetrical about the centre line of the centre span so that stresses due to any symmetrical loading will also be symmetrical about the same point. The foregoing results agree with this conclusion. Solutions involving symmetrical loadings can therefore be simplified. This applies to dead load, live load on span l_1, and temperature and shrinkage stresses.

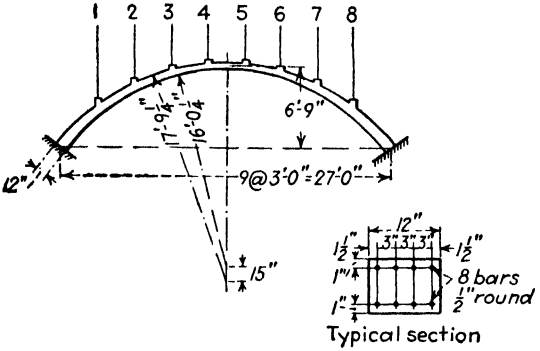
In Table VI if M_2, H_2, and V_2 are set equal to M_0, H_0, and $-V_0$, respectively, but six unknowns remain and of these V_1 evidently equals zero. There remain then but five unknowns to solve and five independent equations only are required for the solution instead of nine since tabular Eqs. (7) to (9) become duplicates of tabular Eqs. (1) to (3), respectively, and may be omitted. The five revised independent equations then become:

TABLE VII

Equation	M_0	$+ \ H_0$	$+ \ V_0$	$+ \ M_1$	$+ \ H_1$	$=$ C (00 omitted)
1	53.52	74.17	− 12.10	− 2.32	− 34.86	2,007.0
2	74.17	1,036.0	−301.8	−33.71	−619.57	8,605.0
3	−12.10	− 301.8	13,716.0	88.40	1,330.0	29,381.0
4	− 4.64	− 67.42	176.80	51.95	86.44	1,739.0
5	−69.72	−1,239.14	2,660.0	86.44	1,317.0	4,730.0

These equations are quickly solved simultaneously by the Doolittle method given in U. S. Coast and Geodetic Survey publication listed at end of chapter and produce results identical with the solution previously given by the method which would have to be applied if the structure were not symmetrical and symmetrically loaded.

Problems



Prob. 1.

1. Determine the value of the horizontal and vertical components of the left reaction, also its moment about an axis passing through the intersection of rib axis and springing line, due to a vertical load of unity at each panel point. Assume that under actual loads on the arch the line of resistance always remains within the kernel of the cross section.

2. A symmetrical, no-hinged, filled-spandrel, concrete arch has the following properties: span, 140 ft.; rise, 20 ft.; live load, 60 lb. per square foot; earth fill, 100 lb. per cubic foot; depth of fill at crown, 1.5 ft.

- *a.* Determine crown thickness by use of the Weld equation.
- *b.* Determine ordinates to the arch axis at the tenth points, using both the Cochrane and the Whitney equations.
- *c.* Plot the Cochrane and Whitney axes for half the span. Use a scale of 1 in. = 10 ft. 0 in., and plot both curves to the same axes.

3. Show that Eqs. (25), (26), and (27) agree with Eqs. (38), (37), and (34) of Art. 45 when applied to a single span with rigid abutments if effect of direct work is neglected, the arch rib divided into sections such that $\Delta s/I$ is constant and signs of m_L and m_R changed to correspond to nomenclature used in developing the latter equations.

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CHAPTER IX

RIGID FRAMES

91. General.—The ordinary arch bridge is not especially well adapted in either appearance or convenience for overhead highway crossings where limited headroom only is available, and in recent years reinforced-concrete and steel structures of the so-called rigid-frame type have been widely used for such bridges. In these structures the transverse members can be made considerably thinner and flatter than the usual arch rib, thus increasing the underclearance somewhat at the centre and considerably at the abutments. The increased headroom is particularly economical in grade-crossing elimination projects where filled approaches are required. It also permits the combination in one structure of the abutments and arch ring which is oftentimes advantageous. This type is illustrated by Fig. 1 and also by the photograph of the West Street Bridge, Dedham, Mass., shown by Plate XII.

The development of the rigid-frame bridge is largely due to the engineering staff of the Westchester, New York, County Commission, and Arthur G. Hayden, designing engineer of the commission, is entitled to special credit for his thorough investigations of the feasibility and economy of this type of structure.

The rigid-frame bridge may be either fixed or hinged at the ends, the former type being more economical if foundation conditions permit its use.

92. Theory.—Inasmuch as a rigid-frame structure has the same characteristics as an arch, the external moment, shear, and thrust at the centre of the transverse member may be computed by the methods previously developed for arches, as shown in the following articles.

93. Distribution of Internal Stress at Corners.—The internal-stress conditions at the connections between the side walls and the transverse member of a rigid frame are complicated and have been the subject of numerous experimental investigations. The

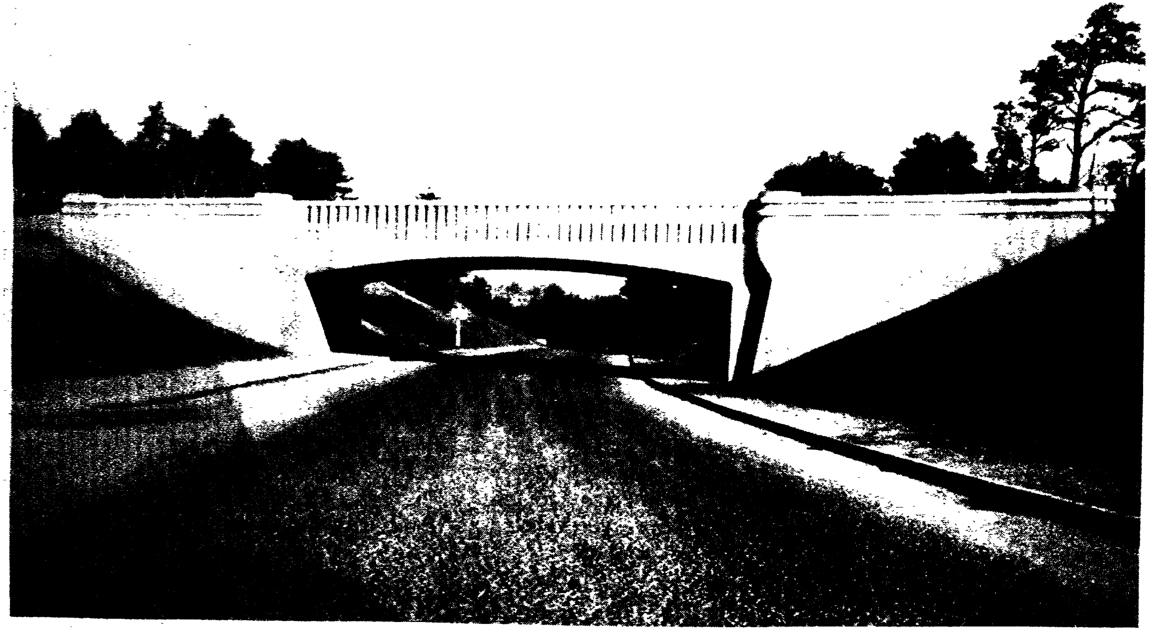


Plate XII.—West Street Bridge. Dedham, Mass.

(*Facing page* 216.)

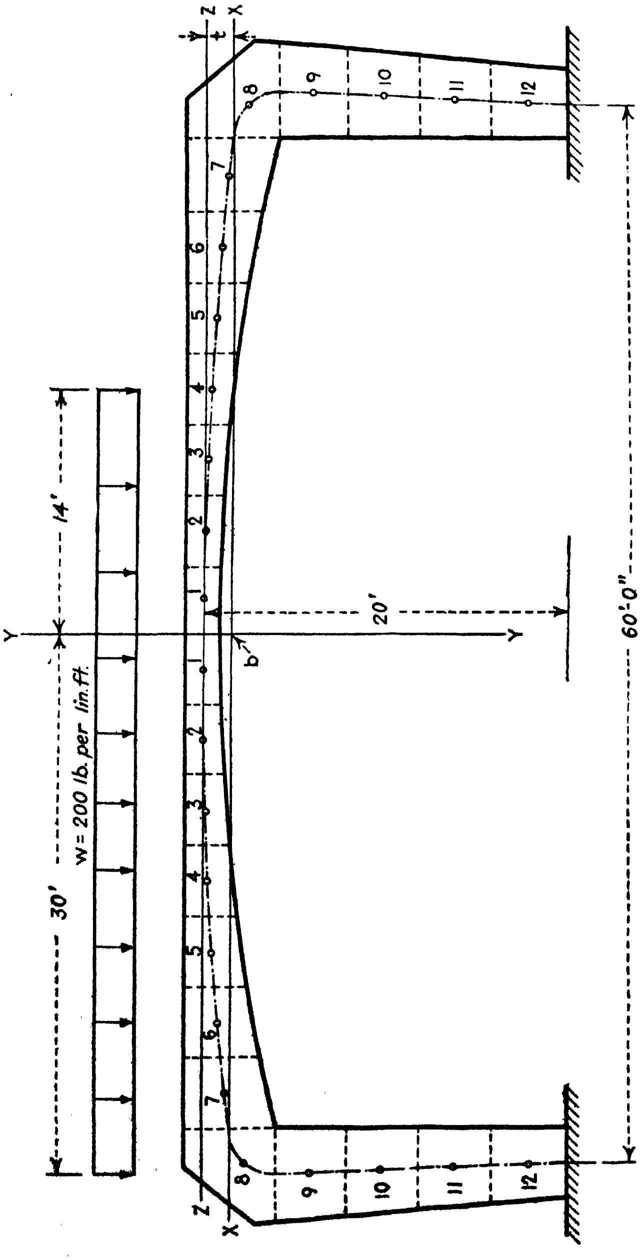


FIG. 1.—Case *A*: uniform load over portion of span.

results of some such investigations are covered in certain of the references given at the end of this chapter.

Figures 2 to 5 show the distribution of stresses in such structures as determined in a bakelite model by the use of polarized light by A. R. Anderson, S.M., Massachusetts Institute of Technology, 1935. The curves show the approximate top- and bottom-edge stresses in pounds per square inch. The effect of a centre load on a round-ended symmetrical frame with and without fillets at corners is shown by Figs. 2 and 3 and the effect of a single load at one quarter point only on a fixed-ended frame is shown by Figs. 4 and 5. The figures show the concentration of stress both under the concentrated loads and also at the sharp corners of the unfilleted frames. It should also be noted that the neutral axis at the corners is pulled in toward the inside of the frame in all cases but that this effect is much more pronounced in the cases of the sharp corner. It is also of interest that at a

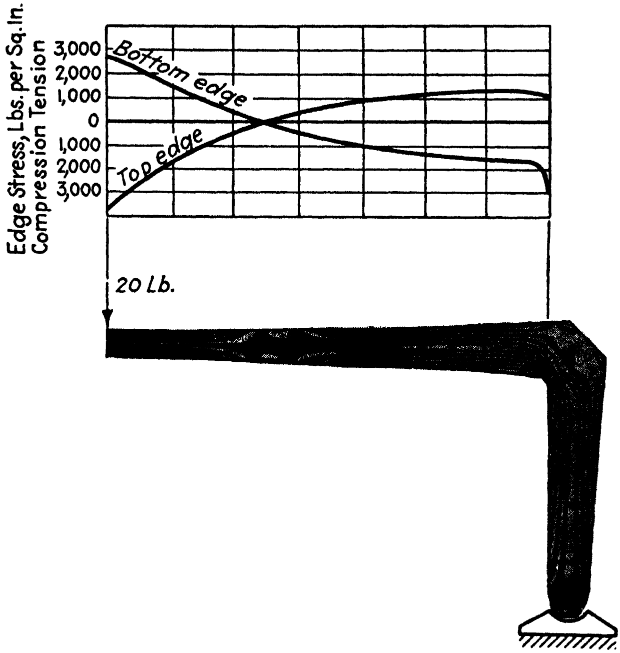


FIG. 2.

corner the stress at the outside edge of the frame is relatively small as compared with the stress at the inside edge.

94. Method of Solution for Symmetrical Fixed-ended Rigid Frame.—In the solution that follows, the unknowns will be taken as M_0, H_0, and V_0 acting at point b as shown in Fig. 6 in which the outer forces applied to the frame are also indicated.

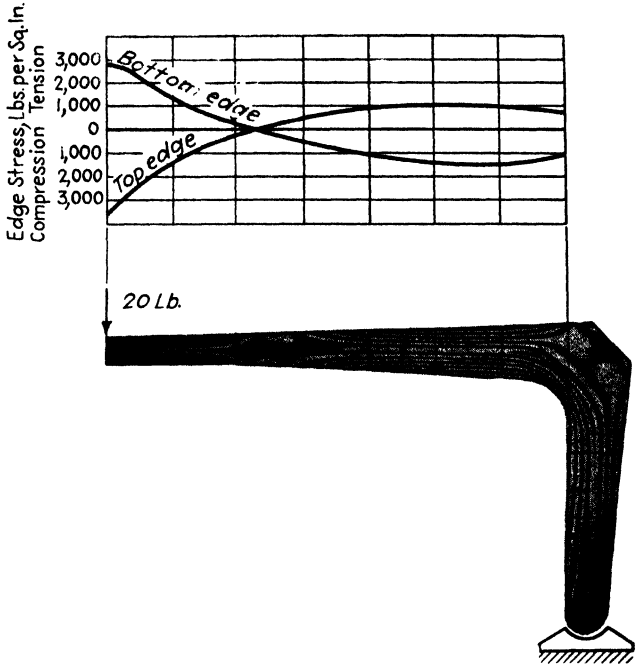


FIG. 3.

Point b is located on a vertical line through the mid-span point at an elevation such that $\int y\frac{ds}{EI} = 0$. The equations for M_0, H_0, and V_0 will be derived by the method of least work.

Let x and y = coordinates of any point on the axis of the frame referred to the axes XX and YY, x being positive either to the left or to the right of the axis YY, and y being positive above the axis XX.

z = distance from axis ZZ, Fig. 1, to any point on the axis of the frame being positive when downward.

M_L = actual moment about an axis normal to frame passing through any point x, y on left half of frame positive when counterclockwise.

M_R = actual moment about a corresponding axis on right half of frame positive when clockwise.

T_{LG} = direct thrust upon a normal section of frame passing through point x, y on left half of girder.

T_{RG} = direct thrust at a corresponding section on right half of girder.

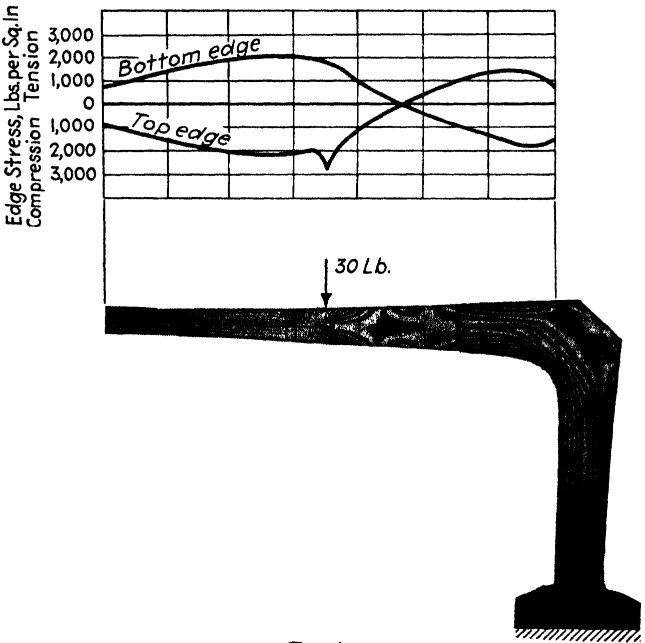


FIG. 4.

T_{LC} = direct thrust upon any normal section of frame passing through any point x, y on left column of frame.

T_{RC} = direct thrust at a corresponding section on right column of frame.

s = length in feet of any portion of the axis of the frame.

m_L = numerical value of moment about an axis normal to frame passing through any point x, y on left half of frame due to loads applied between that point and b, assuming left half of frame to be cantilevered from the left abutment.

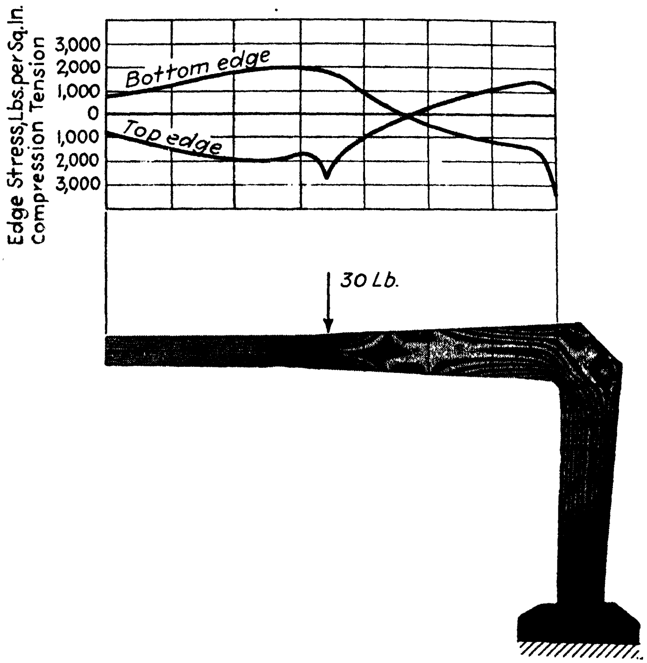


FIG. 5.

m_R = similar moment about a corresponding axis on the right half of the frame.

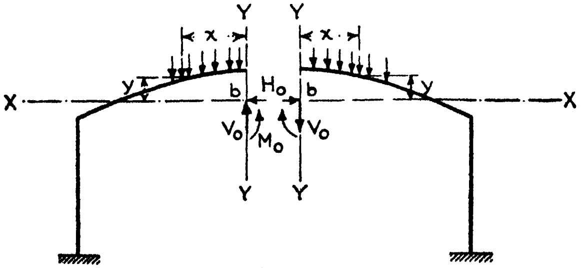


FIG. 6.

w_L = sum of applied vertical loads between b and any point x, y of left column.

w_R = same as w_L but for right column.
I = moment of inertia in feet units of a normal cross section of the frame about an axis normal to the frame at point x, y.
A = area in square feet of a normal cross section of the frame at point x, y.
E = modulus of elasticity in feet units.
W = internal work in entire frame.

Then

$$W = \int M_L^2 \frac{ds}{2EI} + \int M_R^2 \frac{ds}{2EI} + \int T_{LG}^2 \frac{ds}{2AE} + \int T_{RG}^2 \frac{ds}{2AE} + \int T_{LC}^2 \frac{ds}{2AE} + \int T_{RC}^2 \frac{ds}{2AE}$$

where

$$\begin{aligned} M_L &= M_0 + V_0x - H_0y - m_L \\ M_R &= M_0 - V_0x - H_0y - m_R \end{aligned}$$

and, since axis of girder is nearly horizontal, and columns are nearly vertical, assume for the thrust terms that

$$\begin{aligned} T_{LG} &= H_0 \\ T_{RG} &= H_0 \\ T_{LC} &= w_L - V_0 \\ T_{RC} &= w_R + V_0 \end{aligned}$$

This assumption in connection with the thrust terms is justified, since the effect of the thrust on the unknowns M_0, H_0, and V_0 is small.

Now equations for M_0, H_0, and V_0 may be obtained from the conditions that $\partial W/\partial M_0 = 0$, $\partial W/\partial H_0 = 0$, and $\partial W/\partial V_0 = 0$.

In the following equations $\int^L$ means that the integration is made over the entire left half of the frame, $\int^R$ over the entire right half of the frame, $\int^{LC}$ over the left column, $\int^{RC}$ over the right column, $\int^{LG}$ over the left half of the girder, and $\int^{RG}$ over the right half of the girder.

$$\frac{\partial W}{\partial M_0} = \int^L (M_0 + V_0x - H_0y - m_L)\frac{ds}{EI} + \int^R (M_0 - V_0x - H_0y - m_R)\frac{ds}{EI} = 0$$

from which, if E is constant throughout, and since

$$\int^L \frac{ds}{I} = \int^R \frac{ds}{I} \quad \text{and} \quad \int \frac{yds}{I} = 0$$

$$M_0 = \frac{\int^L m_L\frac{ds}{I} + \int^R m_R\frac{ds}{I}}{2\int^L \frac{ds}{I}} \tag{1}$$

If integration is replaced by summation and the structure divided into parts of equal length Δs, Eq. (1) becomes

$$M_0 = \frac{\sum^L \frac{m_L}{I} + \sum^R \frac{m_R}{I}}{2\sum^L \frac{1}{I}} \tag{2}$$

In the same way an equation for V_0 may be derived.

$$\frac{\partial W}{\partial V_0} = \int^L (M_0 + V_0x - H_0y - m_L)\frac{xds}{EI} + \int^R (M_0 - V_0x - H_0y - m_R)(-x)\frac{ds}{EI} + \int^{LC} (w_L - V_0)\frac{(-1)ds}{AE} + \int^{RC} (w_R + V_0)\frac{ds}{AE} = 0$$

from which, if E is constant throughout, and since

$$\int^L \frac{x^2ds}{I} = \int^R \frac{x^2ds}{I} \quad \text{and} \quad \int^{LC} \frac{ds}{A} = \int^{RC} \frac{ds}{A},$$

$$V_0 = \frac{\int^L m_L\frac{xds}{I} - \int^R m_R\frac{xds}{I} + \int^{LC} w_L\frac{ds}{A} - \int^{RC} w_R\frac{ds}{A}}{2\int^L \frac{x^2ds}{I} + 2\int^{LC} \frac{ds}{A}} \tag{3}$$

If integration is replaced by summation, and the structure divided into parts of equal length Δs, Eq. (3) becomes

$$V_0 = \frac{\sum^{L}\frac{m_L x}{I} - \sum^{R}\frac{m_R x}{I} + \sum^{LC}\frac{w_L}{A} - \sum^{RC}\frac{w_R}{A}}{2\sum^{L}\frac{x^2}{I} + 2\sum^{LC}\frac{1}{A}} \tag{4}$$

The derivation of the equation for H_0 follows.

$$\frac{\partial W}{\partial H_0} = \int^{L}(M_0 + V_0 x - H_0 y - m_L)(-y)\frac{ds}{EI} + \int^{R}(M_0 - V_0 x - H_0 y - m_R)(-y)\frac{ds}{EI} + \int^{LG}(H_0)\frac{ds}{AE} + \int^{RG}(H_0)\frac{ds}{AE} = 0$$

from which, if E is constant, and since

$$\int^{L}\frac{y^2 ds}{I} = \int^{R}\frac{y^2 ds}{I} \quad \text{and} \quad \int^{LG}\frac{ds}{A} = \int^{RG}\frac{ds}{A},$$

$$H_0 = \frac{-\int^{L} m_L y\frac{ds}{I} - \int^{R} m_R y\frac{ds}{I}}{2\int^{L} y^2\frac{ds}{I} + 2\int^{LG}\frac{ds}{A}} \tag{5}$$

If integration is replaced by summation and the structure divided into parts of equal length Δs, Eq. (5) becomes

$$H_0 = \frac{-\sum^{L} m_L\frac{y}{I} - \sum^{R} m_R\frac{y}{I}}{2\sum^{L}\frac{y^2}{I} + 2\sum^{LG}\frac{1}{A}} \tag{6}$$

95. Problem: Determine the values of M_0, H_0, and V_0 for the fixed-ended reinforced-concrete barrel, rigid frame shown in Fig. 1. The analysis is made on a 1-ft. strip of the frame normal to the plane of the paper. The axis of the frame is divided into sections of equal length Δs; in this case Δs is 4 ft. The sections are numbered as shown in Fig. 1.

Case A. For a vertical load of 200 lb. per foot applied as shown in Fig. 1.

Case B. For a uniformly varying load applied as shown in Fig. 7.

Solution: Determine first the structural constants of the frame which are independent of the loading and thus apply to both *A* and *B*. These are

TABLE I

Section	z scaled feet	d scaled feet	$I_c = \frac{d^3}{12}$	A_s sq. ft.	$D = \frac{d}{2} - 0.17$	$I_s = 15A_sD^2$	$I = I_c + I_s$	$\frac{1}{I}$	$\frac{z}{I}$	$\frac{1}{A} = \frac{1}{d + 15A_s}$
1	0.00	1.9	0.572	0.020	0.78	0.183	0.755	1.324	0.000	0.455
2	0.1	2.1	0.772	0.020	0.88	0.231	1.003	0.997	0.100	0.417
3	0.25	2.3	1.014	0.018	0.98	0.259	1.273	0.786	0.196	0.389
4	0.4	2.7	1.640	0.015	1.18	0.313	1.953	0.512	0.205	0.342
5	0.6	3.2	2.731	0.015	1.43	0.460	3.191	0.313	0.188	0.292
6	1.0	3.9	4.943	0.018	1.78	0.855	5.798	0.172	0.172	0.240
7	1.4	4.5	7.594	0.020	2.08	1.298	8.892	0.112	0.157	0.208
8	2.5	5.0	10.417	0.020	2.33	1.628	12.045	0.083	0.207	0.189
9	6.0	5.1	11.054	0.020	2.38	1.699	12.753	0.078	0.468	0.185
10	10.0	4.8	9.216	0.018	2.23	1.343	10.559	0.095	0.950	0.197
11	14.0	4.5	7.594	0.020	2.08	1.298	8.892	0.112	1.568	0.208
12	18.0	4.2	6.174	0.020	1.93	1.117	7.291	0.137	2.466	0.222
Σ	......	......	......	......	......	......	....	4.721	6.677	

$$t = \frac{\sum \frac{z}{I}}{\sum \frac{1}{I}} = \frac{6.677}{4.721} = 1.414$$

TABLE II

Section	x scaled feet	$y = t - z$	x^2	y^2	x/I	y/I	x^2/I	y^2/I
1	2	+ 1.41	+4.0	+1.988	+2.65	+1.868	+5.3	+2.634
2	6	+ 1.31	36.0	1.716	5.98	+1.306	35.9	1.711
3	10	+ 1.16	100.0	1.346	7.86	+0.911	78.6	1.057
4	14	+ 1.01	196.0	1.020	7.17	+0.517	100.4	0.522
5	18	+ 0.81	324.0	0.656	5.64	+0.254	101.5	0.205
6	22	+ 0.41	484.0	0.168	3.79	+0.071	83.4	0.029
7	26	+ 0.01	676.0	0.000	2.92	+0.001	75.9	0.000
8	29.8	− 1.09	888.0	1.188	2.47	−0.091	73.6	0.099
9	30.5	− 4.59	930.3	21.068	2.39	−0.359	72.9	1.647
10	30.3	− 8.59	918.1	73.788	2.87	−0.814	87.0	6.992
11	30.1	−12.59	906.0	158.508	3.38	−1.416	101.1	17.827
12	30.0	−16.59	900.0	275.228	4.11	−2.263	123.4	37.543
Σ	....	....	....	....	....	−0.015	939.0	70.266

shown in Tables I and II. The ordinates *z* of any point are scaled from the axis *zz* which is a horizontal axis through the intersection of the axis of the frame and the axis YY. The distance d is the depth of the normal cross section at any point on the frame axis. I_c is the moment of inertia of the concrete area of a normal cross section. A_s is the total area of the reinforcement on both sides of the axis. D is the distance from the frame axis to the steel reinforcement, the cover on the steel being 2 in. I_s is the moment of inertia of the steel area about the frame axis and is taken as nA_sD^2 where n is the ratio of the modulus of elasticity of steel to that of concrete and has a numerical value of 15. The total moment of inertia I of any normal section is the sum of I_c and I_s. Likewise, the total area of any normal section is the sum of $A_c + 15A_s$ where

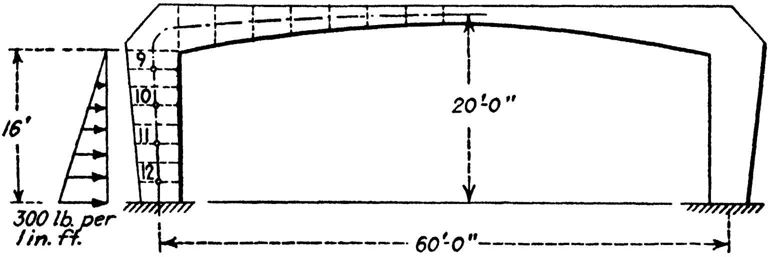


FIG. 7.

TABLE III

Section	m_L kip ft.	m_R kip ft.	(m_L+m_R)	(m_L-m_R)	$(m_L+m_R)\frac{1}{I}$	$(m_L-m_R)\frac{x}{I}$	$(m_L+m_R)\frac{y}{I}$
1	0.4	0.4	0.8	0	1.06	0	1.49
2	3.6	3.6	7.2	0	7.18	0	9.40
3	10.0	10.0	20.0	0	15.72	0	18.23
4	19.6	19.6	39.2	0	20.06	0	20.26
5	32.4	30.8	63.2	1.6	19.78	9.02	16.02
6	48.4	42.0	90.4	6.4	15.54	23.74	6.37
7	67.6	53.2	120.8	14.4	13.52	42.05	0.13
8	88.8	63.8	152.6	25.0	12.66	61.75	− 13.80
9	93.0	65.8	158.8	27.2	12.39	65.01	− 56.87
10	91.8	65.2	157.0	26.6	14.92	76.26	−128.16
11	90.6	64.7	155.3	25.9	17.39	87.61	−218.94
12	90.0	64.4	154.4	25.6	21.15	105.31	−350.87
Σ	....	....	.....	....	+171.37	+470.75	−696.74

$$A_c = 1 \times d = d \text{ sq. ft.}$$

Since point b is located so that $\int y \frac{ds}{I} = 0$, it is evident that the distance t on Fig. 1 is given by the following equation:

$$t = \frac{\sum \frac{z}{I}}{\sum \frac{1}{I}}$$

Table III gives the computation of the load terms for *Case A*. Table IV gives the computation of the load terms for *Case B*.

TABLE IV

Section	m_L kip ft.	m_R kip ft.	$(m_L + m_R)$	$(m_L - m_R)$	$(m_L + m_R)\frac{1}{I}$	$(m_L - m_R)\frac{x}{I}$	$(m_L + m_R)\frac{y}{I}$
1	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0
9	0.025	0	0.025	0.025	0.002	0.06	− 0.009
10	0.675	0	0.675	0.675	0.064	1.94	− 0.549
11	3.125	0	3.125	3.125	0.350	10.56	− 4.425
12	8.575	0	8.575	8.575	1.175	35.24	−19.405
Σ	.....	..	.....	.....	1.591	47.80	−24.388

Using the foregoing tabular values gives the following results:

Case A:

Neglecting effect of direct stress,

$$M_0 = \frac{\sum (m_L + m_R)\frac{1}{I}}{2\sum \frac{1}{I}} = \frac{+171.37}{2(4.721)} = +18.15 \text{ kip ft.}$$

$$V_0 = \frac{\sum (m_L - m_R)\frac{x}{I}}{2\sum \frac{x^2}{I}} = \frac{+470.75}{2(939.0)} = +0.250 \text{ kip}$$

$$H_0 = -\frac{\sum (m_L + m_R)\frac{y}{I}}{2\sum \frac{y^2}{I}} = -\frac{-696.74}{2(70.266)} = +4.95 \text{ kip}$$

M_C = moment at crown $= M_0 - H_0 \times t = +18.15 - (4.95)(1.414) =$
$18.15 - 7.00 = +11.15$ kip ft.

Consider, now, the effect of including the direct-stress terms of the equations. The quantities w_L and w_R for the given loading are constant throughout the columns.

So

$$\sum^{LC}(w_L - w_R)\frac{1}{A} = (w_L - w_R)\sum^{LC}\frac{1}{A}$$

where $(w_L - w_R) = 6000 - 2800 = 3200$ lb. $= 3.2$ kip

$$\sum^{LC}\frac{1}{A} = 0.812 \qquad \sum^{LG}\frac{1}{A} = 2.532$$

Hence,

$$M_0 = +18.15 \text{ kip ft.}$$
$$V_0 = \frac{470.75 + (3.2)(0.812)}{2(939.0 + 0.812)} = +0.252 \text{ kip}$$
$$H_0 = -\frac{696.74}{(2)(70.266 + 2.532)} = +4.78 \text{ kips}$$
$$M_C = +18.15 - (4.78)(1.414) = 18.15 - 6.76 = +11.39 \text{ kip ft.}$$

Case B:

Neglecting effect of direct stress,

$$M_0 = \frac{+1.591}{2(4.721)} = +0.169 \text{ kip ft.}$$
$$V_0 = \frac{47.80}{2(939.0)} = +0.025 \text{ kip}$$
$$H_0 = -\frac{-24.388}{2(70.266)} = +0.173 \text{ kip}$$
$$M_C = +0.169 - 0.173 \times 1.414 = 0.169 - 0.245 = -0.076 \text{ kip ft.}$$

Considering effect of direct stress, noting $w_L = w_R = 0$,

$$M_0 = +0.169 \text{ kip ft}$$
$$V_0 = \frac{47.80}{2(939.0) + 0.812)} = +0.025 \text{ kip}$$
$$H_0 = -\frac{-24.39}{2(70.266 + 2.532)} = +0.167 \text{ kip}$$
$$M_C = +0.169 - 0.167 \times 1.414 = 0.169 - 0.236 = -0.067 \text{ kip ft.}$$

Note that if the same loading were applied acting inward on the right leg in addition to that on left column, V_0 would be zero, and M_0 and H_0 twice the values given above.

96. Temperature Stresses in a Symmetrical Fixed-ended Rigid Frame.—For stresses due to a uniform increase in temperature throughout the entire frame, m_L, m_R, w_L, and w_R each equal zero, hence M_0, V_0 each equal zero so that the only unknown is H_0.

In addition to the notation used in Art. 94,

Let ϵ = coefficient of expansion.

t = change in temperature, positive when an increase.

ϕ = angle between horizontal and the tangent to the axis of the frame at any point x, y of the girder.

The following expression for the internal work due to temperature changes while perhaps not strictly logical from the standpoint of internal work gives results identical with those obtained by the method of deflection in Chap. VIII.

$$W = \int M_L{}^2\frac{ds}{2EI} + \int M_R{}^2\frac{ds}{2EI} + \int T_{LG}{}^2\frac{ds}{2AE} + \int T_{RG}{}^2\frac{ds}{2AE} - \int T_{LG}\epsilon t ds - \int T_{RG}\epsilon t ds$$

Inasmuch as there is no direct stress in columns, no terms for direct stress in columns need be included in the foregoing expression. The signs of the last two terms in the expression for internal work are negative because T_{LG} and T_{RG} have been taken as positive when a thrust, and t is positive when an increase. Now

$$M_L = -H_0 y \qquad \text{and} \qquad M_R = -H_0 y;$$

and

$$T_{LG} = H_0 \cos\phi \qquad \text{and} \qquad T_{RG} = H_0 \cos\phi.$$

The equation for H_0 may be obtained from the condition that $\partial W/\partial H_0 = 0$.

$$\frac{\partial W}{\partial H_0} = \int^{L}(-H_0 y)(-y)\frac{ds}{EI} + \int^{R}(-H_0 y)(-y)\frac{ds}{EI} + \int^{LG}(H_0\cos\phi)(\cos\phi)\frac{ds}{AE} + \int^{RG}(H_0\cos\phi)(\cos\phi)\frac{ds}{AE} - \int^{LG}(\cos\phi)\epsilon t ds - \int^{RG}(\cos\phi)\epsilon t ds = 0$$

Now,

$$\cos\phi ds = dx, \quad \text{so} \quad -\epsilon t\left[\int^{LG}\cos\phi ds + \int^{RG}\cos\phi ds\right] = -\epsilon t L,$$

where L = total span length of girder.

Then, if E is constant, and since $\int^{L} y^2\frac{ds}{I} = \int^{R} y^2\frac{ds}{I}$, and

$$\int^{LG} \cos^2 \phi\frac{ds}{A} = \int^{RG} \cos^2 \phi\frac{ds}{A},$$

$$H_0 = \frac{\epsilon tLE}{2\int^{L} y^2\frac{ds}{I} + 2\int^{LG} \cos^2 \phi\frac{ds}{A}}$$

Since the second term in the denominator is small, and $\cos^2 \phi$ is approximately equal to unity, the equation may be written

$$H_0 = \frac{\epsilon tLE}{2\int^{L} y^2\frac{ds}{I} + 2\int^{LG}\frac{ds}{A}} \tag{7}$$

If integration is replaced by summation, and the structure divided into parts of equal length Δs, Eq. (7) becomes

$$H_0 = \frac{\epsilon tLE}{2\Delta s\left[\sum^{L}\frac{y^2}{I} + \sum^{LG}\frac{1}{A}\right]} \tag{8}$$

If temperature change is a decrease the value of H_0 in Eqs. (7) and 8 will be negative showing tension in the arch rib.

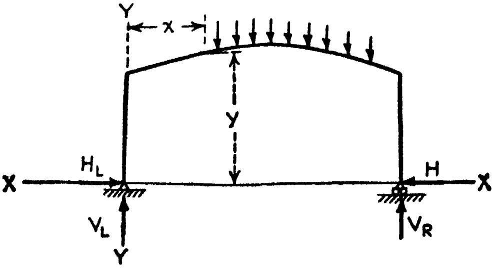


FIG. 8.

97. Method of Solution for Symmetrical Pin-ended Rigid Frame.

—In the development that follows, the unknown will be taken as the horizontal component of the right reaction, as shown in Fig. 8, in which the outer forces applied to the frame are also indicated. The method of least work will be used.

Let x and y = coordinates of any point on the axis of the frame referred to the axes XX and YY, x being positive to the right, and y positive upward.

M = actual moment about an axis normal to frame passing through any point x, y.

M_0 = bending moment about an axis normal to frame passing through any point x, y due to applied loads, right end of frame being assumed to be on rollers.

T_{LC} = direct thrust upon any normal section of frame passing through any point x, y on left column of frame.

T_{RC} = direct thrust at a corresponding section on right column of frame.

T_G = direct thrust at a corresponding section of girder.

W, A, I and E be as defined in Art. 94.

Then

$$W = \int M^2 \frac{ds}{2EI} + \int T_{LC}{}^2 \frac{ds}{2AE} + \int T_{RC}{}^2 \frac{ds}{2AE} + \int T_G{}^2 \frac{ds}{2AE}$$

where

$$M = M_0 - Hy$$

As in Art. 94, assume for the thrust that

$$\begin{aligned} T_G &= H \\ T_{LC} &= V_L \\ T_{RC} &= V_R \end{aligned}$$

The equation for H may now be obtained from the condition that $\partial W/\partial H = 0$. In the following equation $\int^F$ means that the integration is made over the entire frame, and $\int^G$ over the girder only. Then

$$\frac{\partial W}{\partial H} = \int^F (M_0 - Hy)(-y)\frac{ds}{EI} + \int^G (H)\frac{ds}{AE} = 0$$

from which, if E is constant throughout,

$$H = \frac{\int^{F} M_0 y \frac{ds}{I}}{\int^{F} y^2 \frac{ds}{I} + \int^{G} \frac{ds}{A}} \tag{9}$$

If integration is replaced by summation, and the frame divided into parts of equal length Δs, Eq. (9) becomes

$$H = \frac{\sum^{F} M_0 \frac{y}{I}}{\sum^{F} \frac{y^2}{I} + \sum^{G} \frac{1}{A}} \tag{10}$$

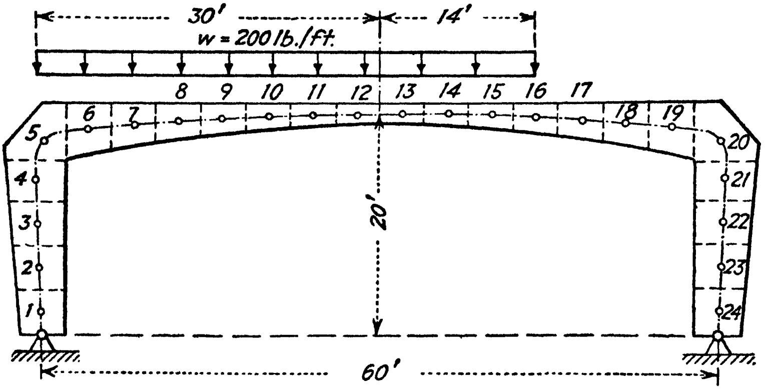


FIG. 9.

98. Problem: Determine the value of H for the pin-ended reinforced-concrete barrel, rigid frame, shown in Fig. 9, which is exactly the same as that used in the example of Art. 95 except that in this case the bottoms of the legs are pinned instead of fixed. The analysis is made on a 1-ft. strip of the frame normal to the plane of the paper. As before, the axis of the frame is divided into sections of equal length Δs, each section having a length of 4 ft. The sections are numbered as shown in Fig. 9.

Solution: The distances x and y were scaled on Fig. 9 and tabulated in Table V. The values of $1/I$ and $1/A$ were taken from Table I. The value of the vertical reactions are those for an end-supported beam. The rest of the calculations of Table V are self-explanatory.

Considering effect of direct stress:

$$H = \frac{11{,}075.34}{3402.36 + 5.06} = 3.250 \text{ kips}$$

Neglecting effect of direct stress:

$$H = \frac{11{,}075.54}{3402.36} = 3.25 \text{ kips}$$

Then using the latter value of H, the moment at the crown, since $V_R = 3.23$, is given by the following expression.

$$M_C = (3.23)(30) - (3.25)(20) - \frac{(0.2)(14)^2}{2} = +12.3 \text{ kip ft.}$$

TABLE V

Section	x	y	$1/I$	$1/A$	y^2/I	M_0	M_0y/I
1	0	+2	0.137	.....	0.55	0	0
2	− 0.1	6	0.112	.....	4.03	− 0.5	− 0.34
3	− 0.3	10	0.095	.....	9.50	− 1.7	− 1.61
4	− 0.5	14	0.078	.....	15.29	− 2.8	− 3.06
5	+ 0.2	17.5	0.083	0.189	25.42	+ 1.1	+ 1.60
6	+ 4	18.6	0.112	0.208	38.75	+20.7	+ 43.12
7	+ 8	19.0	0.172	0.240	62.09	+38.4	+ 125.49
8	+12	19.4	0.313	0.292	117.80	+52.5	+ 318.78
9	+16	19.6	0.512	0.342	196.69	+63.9	+ 641.24
10	+20	19.75	0.786	0.389	306.59	+71.5	+ 1,109.89
11	+24	19.9	0.997	0.417	394.82	+76.2	+ 1,511.83
12	+28	20	1.324	0.455	529.60	+77.4	+ 2,061.06
13	+32	20	1.324	0.455	529.60	+76.0	+ 2,018.56
14	+36	19.9	0.997	0.417	394.82	+71.1	+ 1,410.65
15	+40	19.75	0.786	0.389	306.59	+62.9	+ 976.39
16	+44	19.6	0.512	0.342	196.69	+51.6	+ 517.80
17	+48	19.4	0.313	0.292	117.80	+38.7	+ 234.99
18	+52	19.0	0.172	0.240	62.09	+25.8	+ 84.31
19	+56	18.6	0.112	0.208	38.75	+12.9	+ 26.87
20	+59.8	17.5	0.083	0.189	25.42	+ 0.6	+ 0.87
21	+60.5	14	0.078	.....	15.29	− 1.6	− 1.75
22	+60.3	10	0.095	.....	9.50	− 1.0	− 0.95
23	+60.1	6	0.112	.....	4.03	− 0.3	− 0.20
24	+60	2	0.137	.....	0.55	0	0
Σ	......	.....	.....	5.064	3,402.26	......	+11,075.54

99. Temperature Stresses in a Symmetrical Pin-ended Rigid Frame.—An equation for the unknown H will be derived for a uniform increase in temperature throughout the frame. In this case M_0, V_L, and V_R will all be zero.

The notation will be the same as that previously used. Proceeding as in Art. 96,

$$W = \int M^2 \frac{ds}{2EI} + \int T_G{}^2 \frac{ds}{2AE} - \int T_G \epsilon t ds$$

where $M = -Hy$ and $T_G = H \cos \phi$.

The equation for H may be obtained from the condition that $\partial W/\partial H = 0$.

$$\frac{\partial W}{\partial H} = \int^F (-Hy)(-y)\frac{ds}{EI} + \int^G (H \cos \phi) \cos \phi \frac{ds}{AE} - \int^G \cos \phi \epsilon t ds = 0$$

But $\cos \phi ds = dx$, $\int^G \cos \phi ds = L$, hence if E is constant,

$$H = \frac{\epsilon t L E}{\int^F y^2 \frac{ds}{I} + \int^G \cos^2 \phi \frac{ds}{A}}$$

But, since $\cos^2 \phi$ is small,

$$H = \frac{\epsilon t L E}{\int^F y^2 \frac{ds}{I} + \int^G \frac{ds}{A}} \tag{11}$$

If integration is replaced by summation, and the frame divided in parts of equal length Δs, Eq. (11) becomes

$$H = \frac{\epsilon t L E}{\Delta s \left[\sum^F \frac{y^2}{I} + \sum^G \frac{1}{A} \right]} \tag{12}$$

100. Rigid Frames. Miscellaneous.—The exact analysis of a skewed rigid frame is quite involved and will not be covered in the text, as it has been fully treated by Hayden (see reference at end of chapter). It should be observed that where the skew is not large, the structure may be safely analyzed as a bridge without skew having a span equal to the skew span of the structure using a small amount of transverse reinforcement to take care of torsion.

The analysis of an unsymmetrical fixed-ended frame may be made in the same manner as suggested in Art. 88. A continuous rigid frame is the same kind of structure as a continuous arch on elastic supports and may be analyzed as suggested in Art. 89.

The analysis of an unsymmetrical pin-ended frame may be handled in the same manner as an unsymmetrical two-hinged arch in case *B*, Chap. IV.

Problems

1. Construct an influence line for moment at section (4) on left half of structure shown in Fig. 1 and determine from it the maximum moment at the section due to a uniform live load applied at top of 1000 lb. per linear foot and a single concentrated load of 20,000 lb. Plot positive moment above axis. Neglect effect of direct stress.

2. Same as Prob. 1 but apply to section (9) of structure shown in Fig. 9.

References

The Rigid Frame Bridge, Hayden. John Wiley & Sons, Inc., New York. 1931.

Continuous Frames of Reinforced Concrete, Cross and Morgan. John Wiley & Sons, Inc., New York. 1932.

Progress Reports Nos. 1 and 2 on Stress Distribution in Steel Rigid Frames. National Bureau of Standards in cooperation with American Institute of Steel Construction.

Comments on Rigid Frame Bridges, Hayden. *Civil Eng.*, May, 1936, p. 324.

Specifications for Rigid Frame Bridges, Proceedings, American Railway Eng. Assoc. Chicago. Dec., 1936.

APPENDIX A

CHANGE OF REACTIONS OF CONTINUOUS TRUSSES DUE TO SETTLEMENT OF SUPPORTS AS DETERMINED BY EQUATING EXTERNAL AND INTERNAL WORK

The method of equating external and internal work used in developing expressions for deflection of any point in a truss may be extended as follows to include the effect of the settlement of a point of support of a continuous truss.[1]

Let δ_n = deflection in any direction of any point n of a truss due to settlement of any pier.

s = stress in any truss bar due to force unity acting at point n and in direction of deflection δ_n.

ΔL = change in length of any truss bar.

Δ_m = settlement of any point of support.

R_m = upward reaction at support at which settlement = Δ_m, due to load unity acting at point n and in direction of desired deflection.

S = stress in any bar due to settlement Δ_m of point of support.

Then by equating external and internal work we get the following expression for δ_n:

$$-R_m\Delta_m + 1(\delta_n) = \sum s\frac{SL}{AE} \tag{1A}$$

Consider, now, case A Art. 13. For this case,

$$R_m = -\frac{h}{L_1}, \qquad \delta_n = \delta_a, \qquad \Delta_m = \Delta_0, \qquad s = S_1$$

and

$$S = XS_1$$

Therefore

$$\delta_a = \sum^{\text{Span 6}} X\frac{S_1^2L}{AE} - \frac{\Delta_0 h}{L_1}$$

as previously determined in Art. 13.

---

[1] This method was suggested by C. H. Norris, Jr., S. M., Massachusetts Institute of Technology, 1932.

Equations for deflection of points b, c, and d will be identical with those in Art. 13, *viz.*, Eqs. (36) and (37), since the other supports are assumed not to settle.

For case B (Art. 13):

$$\Delta_m = \Delta_{10}, \qquad R_m = R_{10} = +\frac{h}{L_1} \text{ for span 6}$$

$$R_m = R_{10} = +\frac{h}{L_2} \text{ for span 7}$$

Equation (1A) gives following values:

For span 6:

$$\delta_a = \sum \frac{sSL}{AE} + \frac{\Delta_{10}h}{L_1} = \sum \frac{S_1{}^2XL}{AE} + \frac{\Delta_{10}h}{L_1}$$

For span 7:

$$\delta_b = \sum \frac{sSL}{AE} + \frac{\Delta_{10}h}{L_2} = \sum \frac{S_1{}^2XL}{AE} + \sum \frac{S_1S_2YL}{AE} + \frac{\Delta_{10}h}{L_2}$$

Therefore

$$E(\delta_a + \delta_b) = \Delta_{10}hE\left(\frac{1}{L_1} + \frac{1}{L_2}\right) + aX + bY = 0$$

For δ_c, $R_{10} = -\frac{h}{L_2}$.

Therefore, from Eq. (1A),

For span 7:

$$\delta_c = \sum \frac{sSL}{AE} - \frac{h}{L_2}\Delta_{10} = \sum \frac{S_1S_2L}{AE}X + \sum \frac{S_2{}^2L}{AE}Y - \frac{h}{L_2}\Delta_{10}$$

Also, for span 8:

$$\delta_d = \sum \frac{sSL}{AE} = \sum \frac{S_2{}^2L}{AE}Y$$

Therefore

$$E(\delta_c + \delta_d) = 0 = -\frac{h}{L_2}\Delta_{10}E + bX + aY$$

The foregoing equations agree with those developed in case B (Art. 13).

APPENDIX B

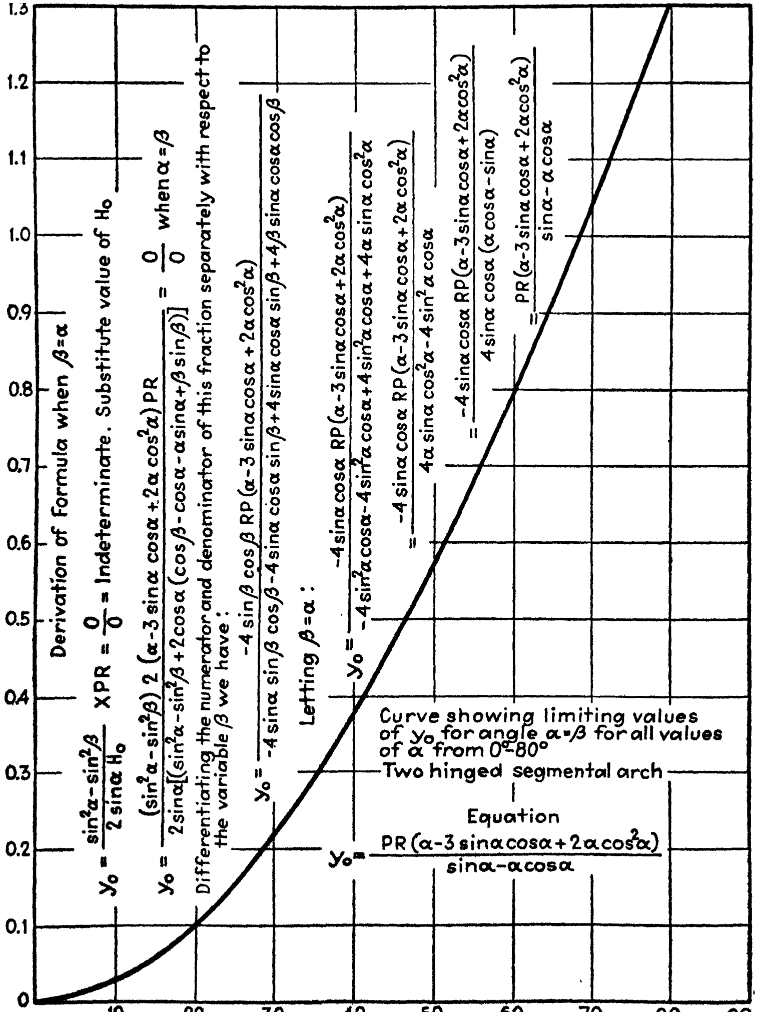
DEVELOPMENT OF EQ. (3) (CHAP. VI) "TWO-HINGED SEGMENTAL ARCHES"

$$
\begin{aligned}
-I_0E\Delta x_a = {} & -(1+K)\int_\beta^\alpha [V_LR(\sin\alpha - \sin\phi) \\
& \qquad + HR(\cos\phi - \cos\alpha)]R(\cos\phi - \cos\alpha)Rd\phi \\
& + KR\int_\beta^\alpha [V_LR(\sin\alpha - \sin\phi) + HR(\cos\phi - \cos\alpha)]R\cos\phi d\phi \\
& + KR\int_\beta^\alpha (-V_L\sin\phi + H\cos\phi)R(\cos\phi - \cos\alpha)Rd\phi \\
& - KR^2\int_\beta^\alpha (-V_L\sin\phi + H\cos\phi)(R\cos\phi)d\phi \\
& - (1+K)\int_{-\alpha}^\beta [V_RR(\sin\alpha + \sin\phi) + \\
& \qquad HR(\cos\phi - \cos\alpha)]R(\cos\phi - \cos\alpha)Rd\phi \\
& + KR\int_{-\alpha}^\beta [V_RR(\sin\alpha + \sin\phi) + \\
& \qquad HR(\cos\phi - \cos\alpha)]R\cos\phi d\phi \\
& + KR\int_{-\alpha}^\beta (V_R\sin\phi + H\cos\phi)R(\cos\phi - \cos\alpha)Rd\phi \\
& - KR^2\int_{-\alpha}^\beta (V_R\sin\phi + H\cos\phi)R\cos\phi d\phi
\end{aligned}
$$

Simplify, noting that $-\int_\beta^\alpha = +\int_\alpha^\beta$ and $-\int_{-\alpha}^\beta = +\int_\beta^{-\alpha}$

$$
\begin{aligned}
-\frac{I_0E\Delta x_a}{R^3} = {} & V_L\int_\alpha^\beta (\sin\alpha - \sin\phi)(\cos\phi - \cos\alpha)d\phi \\
& + KV_L\int_\alpha^\beta (\sin\alpha - \sin\phi)(-\cos\alpha)d\phi - KV_L\int_\alpha^\beta \sin\phi\cos\alpha d\phi \\
& + V_R\int_\beta^{-\alpha} (\sin\alpha + \sin\phi)(\cos\phi - \cos\alpha)d\phi \\
& + KV_R\int_\beta^{-\alpha} (\sin\alpha + \sin\phi)(-\cos\alpha)d\phi - \\
& \qquad KV_R\int_\beta^{-\alpha} (\sin\phi)(-\cos\alpha)d\phi \\
& + H\int_\alpha^\beta (\cos\phi - \cos\alpha)^2d\phi \\
& + HK\int_\alpha^\beta (\cos\phi - \cos\alpha)(-\cos\alpha)d\phi \\
& - HK\int_\alpha^\beta [\cos\phi(\cos\phi - \cos\alpha) - \cos^2\phi]d\phi +
\end{aligned}
$$

$$H\int_{\beta}^{-\alpha}(\cos\phi - \cos\alpha)^2 d\phi$$
$$+ HK\int_{\beta}^{-\alpha}(\cos\phi - \cos\alpha)(-\cos\alpha)d\phi -$$
$$HK\int_{\beta}^{-\alpha}[\cos\phi(\cos\phi - \cos\alpha) - \cos^2\phi]d\phi$$



Ordinates equal Y_0/PR. Abscissas equal α in degrees

FIG. 1-B.

whence

$$-\frac{I_0E\Delta x_a}{R^3} = V_L\int_\alpha^\beta (\sin\alpha - \sin\phi)(\cos\phi - \cos\alpha)d\phi - KV_L\int_\alpha^\beta \sin\alpha\cos\alpha d\phi$$
$$+ V_R\int_\beta^{-\alpha}(\sin\alpha + \sin\phi)(\cos\phi - \cos\alpha)d\phi + KV_R\int_\beta^{-\alpha} - \sin\alpha\cos\alpha d\phi$$
$$+ H\int_\alpha^\beta(\cos\phi - \cos\alpha)^2 d\phi + HK\int_\alpha^\beta \cos^2\alpha d\phi$$
$$+ H\int_\beta^{-\alpha}(\cos\phi - \cos\alpha)^2 d\phi + HK\int_\beta^{-\alpha} \cos^2\alpha d\phi$$

whence

$$-\frac{I_0E\Delta x_a}{R^3} = -(1+K)V_L\sin\alpha\cos\alpha\int_\alpha^\beta d\phi + V_L\sin\alpha\int_\alpha^\beta \cos\phi d\phi + V_L\cos\alpha\int_\alpha^\beta \sin\phi d\phi$$
$$- V_L\int_\alpha^\beta \sin\phi\cos\phi d\phi - (1+K)V_R\sin\alpha\cos\alpha\int_\beta^{-\alpha} d\phi$$
$$+ V_R\sin\alpha\int_\beta^{-\alpha}\cos\phi d\phi - V_R\cos\alpha\int_\beta^{-\alpha}\sin\phi d\phi + V_R\int_\beta^{-\alpha}\sin\phi\cos\phi d\phi$$
$$+ (1+K)H\cos^2\alpha\int_\alpha^\beta d\phi - 2H\cos\alpha\int_\alpha^\beta\cos\phi d\phi + H\int_\alpha^\beta\cos^2\phi d\phi$$
$$+ (1+K)H\cos^2\alpha\int_\beta^{-\alpha} d\phi - 2H\cos\alpha\int_\beta^{-\alpha}\cos\phi d\phi + H\int_\beta^{-\alpha}\cos^2\phi d\phi$$

But

$$\int_\alpha^\beta d\phi + \int_\beta^{-\alpha} d\phi = \int_\alpha^{-\alpha} d\phi$$

also,

$$\int_\alpha^\beta \cos\phi d\phi + \int_\beta^{-\alpha}\cos\phi d\phi = \int_\alpha^{-\alpha}\cos\phi d\phi$$

and

$$\int_\alpha^\beta \cos^2\phi d\phi + \int_\beta^{-\alpha}\cos^2\phi d\phi = \int_\alpha^{-\alpha}\cos^2\phi d\phi$$

Therefore

$$-\frac{I_0E\Delta x_a}{R^3} = -(1+K)\sin\alpha\cos\alpha\left[V_L\int_\alpha^\beta d\phi + V_R\int_\beta^{-\alpha} d\phi\right]$$
$$+\sin\alpha\left[V_L\int_\alpha^\beta \cos\phi d\phi + V_R\int_\beta^{-\alpha}\cos\phi d\phi\right]$$
$$+\cos\alpha\left[V_L\int_\alpha^\beta \sin\phi d\phi - V_R\int_\beta^{-\alpha}\sin\phi d\phi\right]$$
$$-V_L\int_\alpha^\beta \sin\phi\cos\phi d\phi + V_R\int_\beta^{-\alpha}\sin\phi\cos\phi d\phi$$
$$+(1+K)H\cos^2\alpha\int_\alpha^{-\alpha} d\phi - 2H\cos\alpha\int_\alpha^{-\alpha}\cos\phi d\phi +$$
$$H\int_\alpha^{-\alpha}\cos^2\phi d\phi$$

Whence

$$-\frac{I_0E\Delta x_a}{R^3} = -(1+K)\sin\alpha\cos\alpha[(V_L - V_R)\beta - (V_L + V_R)\alpha]$$
$$+\sin\alpha[(V_L - V_R)\sin\beta - (V_L + V_R)\sin\alpha]$$
$$-\cos\alpha[(V_L + V_R)\cos\beta - (V_L + V_R)\cos\alpha]$$
$$-\frac{\cos(2\alpha)}{4}(V_R + V_L) + \frac{\cos(2\beta)}{4}(V_L + V_R)$$
$$+H\left[\frac{3\sin(2\alpha)}{2} - \alpha - (1+K)(2\alpha)(\cos^2\alpha)\right]$$

But

$$V_L = P\frac{\sin\alpha + \sin\beta}{2\sin\alpha} \qquad \text{and} \qquad V_R = P\frac{\sin\alpha - \sin\beta}{2\sin\alpha}$$

hence

$$V_L - V_R = P\frac{\sin\beta}{\sin\alpha}$$

also,

$$V_L + V_R = P$$

Therefore substituting these values in preceding equations, we obtain

$$-\frac{I_0E\Delta x_a}{R^3} = -P\sin\beta(\beta)(1+K)\cos\alpha + P(\alpha)(1+K)(\sin\alpha\cos\alpha)$$
$$+P\sin^2\beta - P\sin^2\alpha - P\cos\beta\cos\alpha + P\cos^2\alpha$$
$$+\frac{P}{4}(\cos^2\beta - \sin^2\beta - \cos^2\alpha + \sin^2\alpha)$$
$$+H[\tfrac{3}{2}\sin 2\alpha - \alpha - (1+K)(2\alpha)(\cos^2\alpha)]$$

But

$$\frac{P}{4}(\cos^2\beta - \sin^2\beta - \cos^2\alpha + \sin^2\alpha) =$$

$$\frac{P}{4}(1 - 2\sin^2\beta - 1 + 2\sin^2\alpha) = \frac{P(\sin^2\alpha - \sin^2\beta)}{2}$$

Therefore

$$-\frac{I_0 E\Delta x_a}{R^3} = P\left[-\frac{\sin^2\alpha}{2} + \frac{\sin^2\beta}{2} - \cos\beta\cos\alpha + \cos^2\alpha + (1+K)(\cos\alpha)(\alpha\sin\alpha - \beta\sin\beta)\right] + H[3\sin\alpha\cos\alpha - \alpha - (1+K)(2\alpha)(\cos^2\alpha)]$$

Therefore

$$-\frac{I_0 E\Delta x_a}{R^3} = P[(1+K)\cos\alpha(\alpha\sin\alpha - \beta\sin\beta) + \frac{\sin^2\beta}{2} - \frac{\sin^2\alpha}{2} - \cos\beta\cos\alpha + \cos^2\alpha] + H[3\sin\alpha\cos\alpha - \alpha - (1+K)(2\alpha)(\cos^2\alpha)]$$

APPENDIX C

EFFECT OF PLACING $\frac{1-FK}{1+GK}$ = UNITY IN EQ. (8) (CHAP. VI), "TWO-HINGED SEGMENTAL ARCH"

To determine the effect of placing $\frac{1-FK}{1+GK}$ = unity, consider one of the arch ribs of the Longfellow Bridge between Boston and Cambridge. These ribs are actually parabolic, but it is assumed that the cross-section properties would not be materially changed by substituting a segmental arch passing through end hinges and crown center. The rib considered has a span length of 145 ft. and a rise of 20.94 ft. The cross section has a gross area of 56.57 sq. in. and a moment of inertia of 21,970 in.^{4} Other properties are shown in Fig. 1*C*.

Using the value of α, *viz.*, 32°13′ and determining constants by interpolation in the tabular values given in Table I (Chap. VI) give the following results:

β/α	H_0	F	G	$L/2h$	$(L/2h)^2$	$(L/2h)^4$
0	0.3837	1.127	0.3935	3.462	11.985	143.64
0.2	0.3638	1.137				
0.4	0.3073	1.171				
0.6	0.2199	1.233				
0.8	0.1133	1.327				
1.0	0.0000	1.465				

Values in the columns marked H_0, F, and G are to be multiplied by the appropriate values shown in the right-hand columns.

$$K = \frac{I_0}{AR^2} = \frac{21{,}970}{56.57(136 \times 12)^2} = 0.000146$$

Values of β/α and the corresponding values of H_0 and F as determined by interpolation for the arch under consideration are shown in the table on page 245.

Values of H will then be as shown on page 246 under heading H_C. The values under the heading H_P are corresponding values for the parabolic arch used as a basis for the circular arch. See next page for computations.

Panel point	β	β/α	H_0	F	G
0	0	0	$0.3837 \times 3.462 = 1.3284$	$1.127 \times 11.985 = 13.507$	56.522
1	6°07′	0.190	$0.3648 \times 3.462 = 1.2629$	$1.136 \times 11.985 = 13.615$	
2	12°19′	0.382	$0.3124 \times 3.462 = 1.0815$	$1.168 \times 11.985 = 13.998$	
3	18°39′	0.579	$0.2291 \times 3.462 = 0.7931$	$1.237 \times 11.985 = 14.825$	
4	25°15′	0.783	$0.1224 \times 3.462 = 0.4238$	$1.319 \times 11.985 = 15.808$	
5	32°13′	1.000	0.00 0	$1.465 \times 11.985 = 17.558$	

	Values of $\frac{1 - FK}{1 + GK}$*
0	$\frac{1 - 13{,}507 \times 0.000146}{1 + 56.522 \times 0.000146} = \frac{1 - 0.00197}{1.0082} = \frac{0.998}{1.008} = 0.990$
1	$\frac{1 - 13.615 \times 0.000146}{1.0082} = \frac{1 - 0.00199}{1.0082} = \frac{0.998}{1.008} = 0.990$
2	$\frac{1 - 13.998 \times 0.000146}{1.0082} = \frac{1 - 0.00204}{1.0082} = \frac{0.998}{1.008} = 0.990$
3	$\frac{1 - 14.825 \times 0.000146}{1.0082} = \frac{1 - 0.00216}{1.0082} = \frac{0.998}{1.008} = 0.990$
4	$\frac{1 - 15.808 \times 0.000146}{1.0082} = \frac{1 - 0.00231}{1.0082} = \frac{0.998}{1.008} = 0.990$
5	$\frac{1 - 17.558 \times 0.000146}{1.0082} = \frac{1 - 0.00256}{1.0082} = \frac{0.997}{1.008} = 0.989$

---

* Note that this equation has its minimum value for any given arch when F is a maximum, *i.e.*, when $\beta/\alpha = 1.0$. See Table I (Chap. VI) noting that G is constant for any arch.

It is quite evident for the case under consideration, which may be considered a typical example of a two-hinged circular arch, that the value of $\frac{1 - FK}{1 + GK}$ may be taken as constant and equal to the value at the crown and that the latter value may be taken as equal to unity without appreciable error; also that the horizontal component of the reaction is very nearly equal to that for a parabolic arch having same rise, span and cross section.

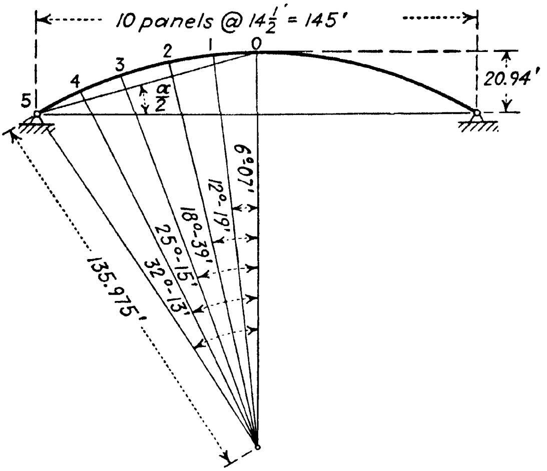


FIG. 1-C.

Panel point	H_C	H_P*
0	1.3284 × 0.99 = 1.3151	1.3370
1	1.2629 × 0.99 = 1.2503	1.2733
2	1.0815 × 0.99 = 1.0707	1.0872
3	0.7931 × 0.99 = 0.7852	0.7941
4	0.4238 × 0.99 = 0.4285	0.4197
5	0	0

K	$-2K^3$	K^4	$K^4 - 2K^3 + K$	H_P
0.1	−0.002	0.0001	0.0981	0.41972
0.2	−0.016	0.0016	0.1856	0.79409
0.3	−0.054	0.0081	0.2541	1.08717
0.4	−0.128	0.0256	0.2976	1.27328
0.5	−0.250	0.0625	0.3125	1.33703

* Values of H_P. Evaluation of constant term:

$$\frac{\frac{Ph}{3}\frac{L}{I_c}}{\frac{1}{A}+\frac{8}{15}\frac{h^2}{I_c}} = P\frac{\frac{(20.94)(145)(12^4)}{3(21970)}}{\frac{(12^2)}{56.57}\times\frac{8}{12}\frac{(20.94)^2(12^4)}{21970}}$$

$$= \frac{955.2528P}{2.5455 \times 220.7227} = \frac{955.2528}{223.2682}P$$

$$= 4.2785P$$

APPENDIX D

DERIVATION OF EQS. (5), (6), (24), AND (25) (CHAP. VII), "DEFLECTION OF TWO-HINGED AND THREE-HINGED ARCH RIBS"

Derivation of Eq. (5) (Chap. VII):

Substituting proper limits in Eq. (4) (Chap. VII) gives the following expressions:

$$
\begin{aligned}
EI_c\delta_{yq} = {} & \int_0^{k_pL}[P(1 - k_p)x - H_py][(1 - k_q)x - H_qy]dx \\
& + \int_{k_pL}^{k_qL}[P(1 - k_p)x - P(x - k_pL) - H_py][(1 - k_q)x - H_qy]dx \\
& + \int_{k_qL}^{L}[Pk_p(L - x) - H_py][k_q(L - x) - H_qy]dx
\end{aligned}
$$

Combining various integrals, noting that certain terms cancel owing to identity of superior and inferior limits, gives the following expression:

$$
\begin{aligned}
EI_c\delta_{yq} = {} & \int_{k_pL}^{k_qL} - P(x - k_pL)[(1 - k_q)x - H_qy]dx \\
& + \int_0^{k_qL}[P(1 - k_p)x - H_py][(1 - k_q)x - H_qy]dx \\
& - \int_0^{k_qL}[Pk_p(L - x) - H_py][k_q(L - x) - H_qy]dx \\
& + \int_0^{L}[Pk_p(L - x) - H_py][k_q(L - x) - H_qy]dx \\
= {} & P\int_0^{k_pL}(x - k_pL)[x - k_qx - H_qy]dx + \\
& \int_0^{k_qL}x[-Px + Pk_pL + Px - Pk_px - H_py]dx \\
& + \int_0^{k_qL}k_qL[-Pk_pL + Pk_px + H_py]dx \\
& + \int_0^{k_qL}[-k_qx - H_qy][-Px + Pk_pL + Px - \\
& \qquad Pk_px - H_py - Pk_pL + Pk_px + H_py]dx \\
& + \int_0^{L}[Pk_p(L - x) - H_py][(k_q(L - x) - H_qy)]dx \\
= {} & P\int_0^{k_pL}[(x - k_pL)][x - k_qx - H_qy]dx +
\end{aligned}
$$

$$\int_0^{k_qL}[Pk_pL - Pk_px - H_py]xdx$$

$$+ \int_0^{k_qL}[Pk_px - Pk_pL + H_py]k_qLdx +$$

$$\int_0^{L}[Pk_p(L - x) - H_py][k_q(L - x) - H_qy]dx$$

$$= \tfrac{1}{6}Pk_pL^3(k_q^3 - 3k_q^2 + 2k_q + k_qk_p^2 - k_p^2) +$$

$$PH_q\Big[k_pL\int_0^{k_pL}ydx$$

$$- \int_0^{k_pL}xydx - k_pL\int_0^{L}ydx + k_p\int_0^{L}xydx\Big] +$$

$$H_p\Big[k_qL\int_0^{k_qL}ydx$$

$$- \int_0^{k_qL}xydx - k_qL\int_0^{L}ydx + k_q\int_0^{L}xydx\Big] +$$

$$H_pH_q\int_0^{L}y^2dx$$

Derivation of Equation (6) (Chap. VII):

For a parabolic arch

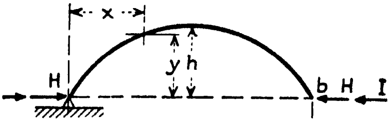


FIG. 1-D.

$$y = \frac{4hx}{L^2}(L - x)$$

Therefore

$$\int ydx = \int\frac{4hx}{L^2}(L - x)dx = \frac{4h}{L^2}\int x(L - x)dx = \frac{4h}{L^2}\Big[\frac{x^2L}{2} - \frac{x^3}{3}\Big]$$

$$\int xydx = \int\frac{4hx^2}{L^2}(L - x)dx = \frac{4h}{L^2}\int x^2(L - x)dx =$$

$$\frac{4h}{L^2}\Big[\frac{x^3L}{3} - \frac{x^4}{4}\Big]$$

$$\int y^2dx = \int\Big(\frac{4hx}{L^2}\Big)^2(L - x)^2dx = \frac{16h^2}{L^4}\int x^2(L - x)^2dx$$

$$= \frac{16h^2}{L^4}\Big[\frac{x^3L^2}{3} - \frac{x^4L}{2} + \frac{x^5}{5}\Big]$$

Introducing appropriate limits and substituting in Eq. (5) (Chap. VII) give

$$\begin{aligned}\delta_{yq} = {}& \frac{Pk_pL^3}{6EI_c}(k_q^3 - 3k_q^2 + 2k_q + k_p^2k_q - k_p^2) + \\ & \frac{PH_q}{EI_c}\left[k_pL\cdot\frac{4h}{L^2}\left(\frac{k_p^2L^3}{2} - \frac{k_p^3L^3}{3}\right)\right. \\ & - \frac{4h}{L^2}\left(\frac{k_p^3L^4}{3} - \frac{k_p^4L^4}{4}\right) - k_pL\cdot\frac{4h}{L^2}\left(\frac{L^3}{2} - \frac{L^3}{3}\right) + \\ & \left. k_p\cdot\frac{4h}{L^2}\left(\frac{L^4}{3} - \frac{L^4}{4}\right)\right] \\ & + \frac{H_p}{EI_c}\left[k_qL\cdot\frac{4h}{L^2}\left(\frac{k_q^2L^3}{2} - \frac{k_q^3L^3}{3}\right) - \frac{4h}{L^2}\left(\frac{k_q^3L^4}{3} - \frac{k_q^4L^4}{4}\right) - \right. \\ & k_qL\cdot\frac{4h}{L^2}\left(\frac{L^3}{2} - \frac{L^3}{3}\right) \\ & \left. + k_q\cdot\frac{4h}{L^2}\left(\frac{L^4}{3} - \frac{L^4}{4}\right)\right] + \frac{H_pH_q}{EI_c}\cdot\frac{16h^2}{L^4}\left(\frac{L^5}{3} - \frac{L^5}{2} + \frac{L^5}{5}\right)\end{aligned}$$

Therefore

$$\begin{aligned}\delta_{yq} = {}& \frac{Pk_pL^3}{6EI_c}(k_q^3 - 3k_q^2 + 2k_q + k_p^2k_q - k_p^2) + \\ & P\,\frac{H_qhk_pL^2}{3EI_c}(2k_p^2 - k_p^3 - 1) \\ & + \frac{H_phk_qL^2}{3EI_c}(2k_q^2 - k_q^3 - 1) + \frac{8}{15}\,\frac{H_pH_qh^2L}{EI_c}\end{aligned}$$

Derivation of Eqs. (24) and (25) (Chap. VII):

$$\delta x_b = \int\frac{Mmdx}{EI_c}$$

But $M = Hm$ and $m = y$
Therefore

$$\delta x_b = H\int\frac{m^2dx}{EI_c} \qquad \text{and} \qquad H = \frac{\delta x_b}{\int m^2dx/EI_c} = \frac{\delta x_b}{\int y^2dx/EI_c}$$

For a parabolic arch,

$$y = \frac{4hx}{L^2}(L - x)$$

and

$$\int y^2 dx = \frac{16h^2}{L^4}\left[\frac{x^3L^2}{3} - \frac{x^4L}{2} + \frac{x^5}{5}\right]$$

Therefore

$$\int_0^L \frac{y^2 dx}{EI_c} = \frac{16h^2}{EI_c}\left(\frac{L}{3} - \frac{L}{2} + \frac{L}{5}\right) = \frac{8}{15}\,\frac{Lh^2}{EI_c}$$

Therefore

$$H = \frac{15}{8}\,\frac{\delta x_b EI_c}{h^2L}$$

APPENDIX E

DERIVATION OF EQS. (25) TO (30), INCLUSIVE (CHAP. VIII), "FIXED-ENDED ARCHES"

Development of Vertical-load Equations.—The equations under consideration are applicable to continuous arches on elastic supports and are given in Art. 89, Chap. VIII. The moment, shear, and thrust at the crown of each span are selected as the redundants in this derivation, and equations for the redundants are developed by the method of least work.

Making the assumptions given in Art. 76, Chap. VIII, the internal work in a bar subject to bending and direct stress is given by the following familiar expression:

$$W = \int \frac{M^2 ds}{2EI} + \int \frac{H_1{}^2 ds}{2AE}$$

where M = total bending moment on a normal section of a bar.
' H_1 = total axial force on a normal section of the bar.

In the development that follows, the notation given in Art. 89 and Fig. 11 (Chap. VIII) is used throughout.

In order to apply the theorem of least work to the system shown in Fig. 11, the expression for total work must be written to include all of the internal work in each member of the structure. This expression will be written in terms of the known loads and the three unknown crown redundants in each span and must then be partially differentiated with respect to each of the independent unknowns giving one independent equation for each unknown. These equations, when set equal to zero and simplified, may then be solved simultaneously, and the crown stresses determined. However, when the expression for total work is written and then partially differentiated with respect to each of the unknown variables, it will be seen that the resulting equations will contain each not more than nine of the unknown crown redundants. These unknown redundants are the three crown redundants in the span under consideration and the three crown redundants in each of the adjacent spans. Therefore, it is possible to develop the three equations for, say, span l_1, without considering any portion of the structure to the left of the crown of l_0 or to the right of the crown of span l_2. These three equations may be called general equations, for they may be applied to preceding and succeeding spans simply by changing subscripts. When the end spans are considered, terms applying to adjacent spans that are nonexistent are set equal to zero. There result, therefore, three equations which may be applied much as the single three-moment equation is applied to successive spans of a continuous beam.

Consider span l_1:

The work in left half of the arch l_1 due to moment is

$$+\int_0^{\frac{S_1}{2}} (M_1 + H_1 y + V_1 x - m_{L_1})^2 \frac{ds}{2EI_1}$$

and in the right half due to moment is

$$+\int_0^{\frac{S_1}{2}} (M_1 + H_1 y - V_1 x - m_{R_1})^2 \frac{ds}{2EI_1}$$

The work in the full span due to direct stress is

$$+2\int_0^{\frac{S_1}{2}} (H_1)^2 \frac{ds}{2EA_1}$$

The work in pier b due to moment is

$$+\int_0^{h_b} \left(M_0 - M_1 + H_0 R_0 - H_1 R_1 - V_0 \frac{l_0}{2} - V_1 \frac{l_1}{2} + H_0 y - \right.$$
$$\left. H_1 y + M_B \right)^2 \frac{ds}{2EI_b}$$

The work in pier b due to direct stress is

$$+\int_0^{h_b} (-W_B - V_0 + V_1)^2 \frac{ds}{2EA_b}$$

The work in pier c due to moment is

$$+\int_0^{h_c} \left(M_1 - M_2 + H_1 R_1 - H_2 R_2 - V_1 \frac{l_1}{2} - V_2 \frac{l_2}{2} + H_1 y - \right.$$
$$\left. H_2 y + M_C \right)^2 \frac{ds}{2EI_c}$$

The work in pier c due to direct stress is:

$$+\int_0^{h_c}(-W_C - V_1 + V_2)^2\frac{ds}{2EA_c}$$

These are the only portions of the expression for total work in which appear the unknowns M_1, H_1, and V_1. When the partial derivatives of the expression for total work $\partial W/\partial M_1$, $\partial W/\partial H_1$, and $\partial W/\partial V_1$ are written, only those terms remain in which M_1, H_1, and V_1 appear. It is unnecessary, therefore, in determining $\partial W/\partial M_1$, $\partial W/\partial H_1$, and $\partial W/\partial V_1$ to consider any parts of the expression for total work that do not contain terms in M_1, H_1, and V_1.

Partial derivatives $\partial W/\partial M_1$, $\partial W/\partial H_1$, and $\partial W/\partial V_1$ may now be written.

$$\begin{aligned}\frac{\partial W}{\partial M_1} = &+\int_0^{\frac{S_1}{2}}(M_1 + H_1y + V_1x - m_{L_1})\frac{ds}{EI_1}\\ &+\int_0^{\frac{S_1}{2}}(M_1 + H_1y - V_1x - m_{R_1})\frac{ds}{EI_1}\\ &+\int_0^{h_b}\Big(M_0 - M_1 + H_0R_0 - H_1R_1 - V_0\frac{l_0}{2} - \frac{V_1l_1}{2}\\ &+ H_0y - H_1y + M_B\Big)(-1)\frac{ds}{EI_b}\\ &+\int_0^{h_c}\Big(M_1 - M_2 + H_1R_1 - H_2R_2 - V_1\frac{l_1}{2} - V_2\frac{l_2}{2}\\ &+ H_1y - H_2y + M_C\Big)\frac{ds}{EI_c}\end{aligned}$$

Integrating where possible and substituting summation for integration along the arch axis, the foregoing equation becomes, if $\frac{\partial W}{\partial M_1} = 0$ and E is constant:

$$\begin{aligned}0 = &M_0\left(-\frac{h_b}{I_b}\right) + H_0\left(-\frac{R_0h_b}{I_b} - \frac{h_b{}^2}{2I_b}\right) + V_0\left(\frac{l_0h_b}{2I_b}\right)\\ &+ M_1\left(2\sum_0^{\frac{S_1}{2}}\frac{\Delta s_1}{I_1} + \frac{h_b}{I_b} + \frac{h_c}{I_c}\right)\end{aligned}$$

$$+ H_1\left(2\sum_0^{\frac{S_1}{2}}\frac{y\Delta s_1}{I_1} + \frac{R_1h_b}{I_b} + \frac{h_b^2}{2I_b} + \frac{R_1h_c}{I_c} + \frac{h_c^2}{2I_c}\right)$$

$$+ V_1\left(\frac{l_1h_b}{2I_b} - \frac{l_1h_c}{2I_c}\right) + M_2\left(-\frac{h_c}{I_c}\right) + H_2\left(-\frac{R_2h_c}{I_c} - \frac{h_c^2}{2I_c}\right)$$

$$+ V_2\left(-\frac{l_2h_c}{2I_c}\right) - \sum_0^{\frac{S_1}{2}}(m_{L_1} + m_{R_1})\frac{\Delta s_1}{I_1} - \frac{M_Bh_b}{I_b} + \frac{M_Ch_c}{I_c} \qquad (25)$$

In a similar manner, partial derivative $\partial W/\partial H_1$, may be written.

$$\frac{\partial W}{\partial H_1} = +\int_0^{\frac{S_1}{2}}(M_1 + H_1y + V_1x - m_{L_1})y\frac{ds}{EI_1}$$

$$+ \int_0^{\frac{S_1}{2}}(M_1 + H_1y - V_1x - m_{R_1})y\frac{ds}{EI_1}$$

$$+ 2\int_0^{\frac{S_1}{2}}\frac{H_1ds}{EA_1} + \int_0^{h_b}\left(M_0 - M_1 + H_0R_0 - H_1R_1 - \frac{V_0l_0}{2} - \frac{V_1l_1}{2}\right.$$

$$\left. + H_0y - H_1y + M_B\right)(-R_1 - y)\frac{ds}{EI_b}$$

$$+ \int_0^{h_c}\left(M_1 - M_2 + H_1R_1 - H_2R_2 - \frac{V_1l_1}{2} - \frac{V_2l_2}{2}\right.$$

$$\left. + H_1y - H_2y + M_C\right)(R_1 + y)\frac{ds}{EI_c}$$

An equation that becomes, if $\partial W/\partial H_1 = 0$ and E is constant,

$$0 = M_0\left(-\frac{R_1h_b}{I_b} - \frac{h_b^2}{2I_b}\right) + H_0\left(-\frac{R_0R_1h_b}{I_b} - \frac{R_1h_b^2}{2I_b} - \frac{R_0h_b^2}{2I_b} - \frac{h_b^3}{3I_b}\right)$$

$$+ V_0\left(\frac{R_1l_0h_b}{2I_b} + \frac{l_0h_b^2}{4I_b}\right) +$$

$$M_1\left(2\sum_0^{\frac{S_1}{2}}\frac{y\Delta s_1}{I_1} + \frac{R_1h_b}{I_b} + \frac{h_b^2}{2I_b} + \frac{R_1h_c}{I_c} + \frac{h_c^2}{2I_c}\right)$$

$$+ H_1\left(2\sum_0^{\frac{S_1}{2}}\frac{y^2\Delta s_1}{I_1} + 2\sum_0^{\frac{S_1}{2}}\frac{\Delta s_1}{A_1} + \frac{R_1^2h_b}{I_b} + \frac{R_1h_b^2}{I_b} + \right.$$

$$\left.\frac{h_b^3}{3I_b} + \frac{R_1^2h_c}{I_c} + \frac{R_1h_c^2}{I_c} + \frac{h_c^3}{3I_c}\right)$$

$$+ V_1\left(\frac{R_1l_1h_b}{2I_b} + \frac{l_1h_b^2}{4I_b} - \frac{R_1l_1h_c}{2I_c} - \frac{l_1h_c^2}{4I_c}\right) + M_2\left(-\frac{R_1h_c}{I_c} - \frac{h_c^2}{2I_c}\right)$$

$$+ H_2\left(-\frac{R_2R_1h_c}{I_c} - \frac{R_2h_c^2}{2I_c} - \frac{R_1h_c^2}{2I_c} - \frac{h_c^3}{3I_c}\right) + V_2\left(-\frac{l_2R_1h_c}{2I_c} - \frac{l_2h_c^2}{4I_c}\right)$$
$$- \sum_0^{\frac{S_1}{2}} (m_{L_1} + m_{R_1})y\frac{\Delta s_1}{I_1} - \frac{M_BR_1h_b}{I_b} - \frac{M_Bh_b^2}{2I_b} + \frac{M_CR_1h_c}{I_c} + \frac{M_Ch_c^2}{2I_c} \qquad (26)$$

In a similar manner, partial derivative $\partial W/\partial V_1$ may be written.

$$\frac{\partial W}{\partial V_1} = \int_0^{\frac{S_1}{2}} (M_1 + H_1y + V_1x - m_{L_1})\frac{x\,ds}{EI_1} + \int_0^{\frac{S_1}{2}} (M_1 + H_1y - V_1x - m_{R_1})(-x)\frac{ds}{EI_1}$$
$$+ \int_0^{h_b}\left(M_0 - M_1 + H_0R_0 - H_1R_1 - \frac{V_0l_0}{2} - \frac{V_1l_1}{2} + H_0y - H_1y + M_B\right)\left(-\frac{l_1}{2}\right)\frac{ds}{EI_b}$$
$$+ \int_0^{h_c}\left(M_1 - M_2 + H_1R_1 - H_2R_2 - \frac{V_1l_1}{2} - \frac{V_2l_2}{2} + H_1y - H_2y + M_C\right)\left(-\frac{l_1}{2}\right)\frac{ds}{EI_c}$$
$$+ \int_0^{h_b} (-W_B - V_0 + V_1)\frac{ds}{EA_b} + \int_0^{h_c} (-W_C - V_1 + V_2)(-1)\frac{ds}{EA_c}$$

An equation that becomes, if $\partial W/\partial V_1 = 0$,

$$0 = M_0\left(-\frac{l_1h_b}{2I_b}\right) + H_0\left(-\frac{R_0l_1h_b}{2I_b} - \frac{l_1h_b^2}{4I_b}\right) + V_0\left(\frac{l_0l_1h_b}{4I_b} - \frac{h_b}{A_b}\right)$$
$$+ M_1\left(\frac{l_1h_b}{2I_b} - \frac{l_1h_c}{2I_c}\right) + H_1\left(\frac{R_1l_1h_b}{2I_b} + \frac{l_1h_b^2}{4I_b} - \frac{l_1R_1h_c}{2I_c} - \frac{l_1h_c^2}{4I_c}\right)$$
$$+ V_1\left(2\sum_0^{\frac{S_1}{2}} \frac{x^2\Delta s_1}{I_1} + \frac{l_1^2h_b}{4I_b} + \frac{l_1^2h_c}{4I_c} + \frac{h_b}{A_b} + \frac{h_c}{A_c}\right) + M_2\left(\frac{l_1h_c}{2I_c}\right)$$
$$+ H_2\left(\frac{R_2l_1h_c}{2I_c} + \frac{l_1h_c^2}{4I_c}\right) + V_2\left(\frac{l_1l_2h_c}{4I_c} - \frac{h_c}{A_c}\right)$$

$$-\sum_0^{\frac{s_1}{2}}(m_{L_1} - m_{R_1})\frac{x\Delta s_1}{I_1} - \frac{M_B l_1 h_b}{2I_b} - \frac{M_C l_1 h_c}{2I_c} - \frac{W_B h_b}{A_B} + \frac{W_C h_c}{A_c} \tag{27}$$

Development of Temperature Equations.—The same procedure is used in the following development as in the development of the vertical-load equations. The expression for internal work due to increase in temperature becomes as follows: (see also Art. 98)

$$W = \int\frac{M^2ds}{2EI} + \int\frac{T^2ds}{2AE} - \int T\epsilon t ds$$

The last term is such that the results by the method of least work check those by the method of deflections.

Instead of the second assumption of Art. 76 it will be assumed that $T_0 = H_0 \cos\phi$, $T_1 = H_1 \cos\phi$, etc., where ϕ is the intersection angle between the tangent to the arch axis at point x,y and the horizontal.

In this analysis the vertical loads will be considered removed so that the resulting equations will give a solution for the redundants due to temperature only and the structure is assumed to be symmetrically heated.

Consider span l_1. The work in the arch is

$$+\int_0^{\frac{s_1}{2}}(M_1 + H_1y + V_1x)^2\frac{ds}{2EI_1} + \int_0^{\frac{s_1}{2}}(M_1 + H_1y - V_1x)^2\frac{ds}{2EI_1}$$
$$+2\int_0^{\frac{s_1}{2}}(-H_1\cos\phi)^2\frac{ds}{2A_1E} + 2\int_0^{\frac{s_1}{2}}(-H_1\cos\phi)\,\epsilon t ds$$

The third term is small, and no serious error is involved by assuming that $\cos^2\phi$ equals unity. In the fourth term

$$ds\cos\phi = dx,$$

so that $2\int_0^{\frac{s_1}{2}}\cos\phi ds = l_1$. Then the work in the arch becomes

$$+\int_0^{\frac{s_1}{2}}(M_1 + H_1y + V_1x)^2\frac{ds}{2EI_1} + \int_0^{\frac{s_1}{2}}(M_1 + H_1y - V_1x)^2\frac{ds}{2EI_1}$$
$$+2\int_0^{\frac{s_1}{2}}(-H_1)^2\frac{ds}{2A_1E} - H_1\epsilon t l_1$$

The work in pier b is

$$+\int_0^{h_b}\left(M_0 - M_1 + H_0R_0 - H_1R_1 - V_0\frac{l_0}{2} - V_1\frac{l_1}{2} + H_0y - H_1y\right)^2\frac{ds}{2EI_b}$$

$$+\int_0^{h_b}(-V_0 + V_1)^2\frac{ds}{2EA_b} + \int_0^{h_b}(-V_0 + V_1)\epsilon t\,ds$$

The work in pier c is

$$+\int_0^{h_c}\left(M_1 - M_2 + H_1R_1 - H_2R_2 - V_1\frac{l_1}{2} - V_2\frac{l_2}{2} + H_1y - H_2y\right)^2\frac{ds}{2EI_c}$$

$$+\int_0^{h_c}(-V_1 + V_2)^2\frac{ds}{2EA_c} + \int_0^{h_c}(-V_1 + V_2)\epsilon t\,ds$$

Whence, the partial derivative $\partial W/\partial M_1$ may be written.

$$\frac{\partial W}{\partial M_1} = +\int_0^{\frac{s_1}{2}}(M_1 + H_1y + V_1x)\frac{ds}{EI_1} + \int_0^{\frac{s_1}{2}}(M_1 + H_1y - V_1x)\frac{ds}{EI_1}$$
$$+\int_0^{h_b}\left(M_0 - M_1 + H_0R_0 - H_1R_1 - V_0\frac{l_0}{2} - V_1\frac{l_1}{2} + H_0y - H_1y\right)(-1)\frac{ds}{EI_b}$$
$$+\int_0^{h_c}\left(M_1 - M_2 + H_1R_1 - H_2R_2 - V_1\frac{l_1}{2} - \frac{V_2l_2}{2} + H_1y - H_2y\right)(1)\frac{ds}{EI_c} = 0$$

Which becomes, when E is constant and summations are used,

$$0 = M_0\left(-\frac{h_b}{I_b}\right) + H_0\left(-\frac{R_0h_b}{I_b} - \frac{h_b^2}{2I_b}\right) + V_0\left(\frac{l_0h_b}{2I_b}\right)$$
$$+ M_1\left(2\sum_0^{\frac{s_1}{2}}\frac{\Delta s_1}{I_1} + \frac{h_b}{I_b} + \frac{h_c}{I_c}\right) +$$
$$H_1\left(2\sum_0^{\frac{s_1}{2}}\frac{y\Delta s_1}{I_1} + \frac{R_1h_b}{I_b} + \frac{h_b^2}{2I_b} + \frac{R_1h_c}{I_c} + \frac{h_c^2}{2I_c}\right)$$

$$+ V_1\left(\frac{l_1h_b}{2I_b} - \frac{l_1h_c}{2I_c}\right) + M_2\left(-\frac{h_c}{I_c}\right) + H_2\left(-\frac{R_2h_c}{I_c} - \frac{h_c^2}{2I_c}\right) + V_2\left(-\frac{l_2h_c}{2I_c}\right) \qquad (28)$$

In a similar manner, partial derivative $\partial W/\partial H_1$ may be written

$$\frac{\partial W}{\partial H_1} = +\int_0^{\frac{S_1}{2}} (M_1 + H_1y + V_1x)\frac{yds}{EI_1} + \int_0^{\frac{S_1}{2}} (M_1 + H_1y - V_1x)\frac{yds}{EI_1} + 2\int_0^{\frac{S_1}{2}} H_1\frac{ds}{EA_1} - \epsilon tl_1$$
$$+ \int_0^{h_b}\left(M_0 - M_1 + H_0R_0 - H_1R_1 - V_0\frac{l_0}{2} - V_1\frac{l_1}{2} + H_0y - H_1y\right)(-R_1 - y)\frac{ds}{EI_b}$$
$$+ \int_0^{h_c}\left(M_1 - M_2 + H_1R_1 - H_2R_2 - V_1\frac{l_1}{2} - V_2\frac{l_2}{2} + H_1y - H_2y\right)(R_1 + y)\frac{ds}{EI_c} = 0$$

Which becomes, when E is constant and summations are used,

$$0 = M_0\left(-\frac{h_bR_1}{I_b} - \frac{h_b^2}{2I_b}\right) + H_0\left(-\frac{R_0R_1h_b}{I_b} - \frac{R_1h_b^2}{2I_b} - \frac{R_0h_b^2}{2I_b} - \frac{h_b^3}{3I_b}\right)$$
$$+ V_0\left(\frac{R_1l_0h_b}{2I_b} + \frac{l_0h_b^2}{4I_b}\right) + M_1\left(2\sum_0^{\frac{S_1}{2}}\frac{y\Delta s_1}{I_1} + \frac{R_1h_b}{I_b} + \frac{h_b^2}{2I_b} + \frac{R_1h_c}{I_c} + \frac{h_c^2}{2I_c}\right)$$
$$+ H_1\left(2\sum_0^{\frac{S_1}{2}}\frac{y^2\Delta s_1}{I_1} + 2\sum_0^{\frac{S_1}{2}}\frac{\Delta s_1}{A_1} + \frac{R_1^2h_b}{I_b} + \frac{R_1h_b^2}{I_b} + \frac{h_b^3}{3I_b} + \frac{R_1^2h_c}{I_c} + \frac{R_1h_c^2}{I_c} + \frac{h_c^3}{3I_c}\right)$$
$$+ V_1\left(\frac{R_1l_1h_b}{2I_b} + \frac{l_1h_b^2}{4I_b} - \frac{R_1l_1h_c}{2I_c} - \frac{l_1h_c^2}{4I_c}\right) + M_2\left(-\frac{R_1h_c}{I_c} - \frac{h_c^2}{2I_c}\right)$$

$$+ H_2\left(-\frac{R_2R_1h_c}{I_c} - \frac{R_2h_c^2}{2I_c} - \frac{R_1h_c^2}{2I_c} - \frac{h_c^3}{3I_c}\right) + V_2\left(-\frac{l_2R_1h_c}{2I_c} - \frac{l_2h_c^2}{4I_c}\right) - \epsilon tl_1E \quad (29)$$

In a similar manner, partial derivative $\partial W/\partial V_1$ may be written.

$$\begin{aligned}
\frac{\partial W}{\partial V_1} = {} & +\int_0^{\frac{S_1}{2}} (M_1 + H_1y + V_1x)\frac{xds}{EI_1} + \int_0^{\frac{S_1}{2}} (M_1 + H_1y - V_1x)(-x)\frac{ds}{EI_1} \\
& + \int_0^{h_b}\left(M_0 - M_1 + H_0R_0 - H_1R_1 - V_0\frac{l_0}{2} - V_1\frac{l_1}{2} + H_0y - H_1y\right)\left(-\frac{l_1}{2}\right)\frac{ds}{EI_b} \\
& + \int_0^{h_c}\left(M_1 - M_2 + H_1R_1 - H_2R_2 - V_1\frac{l_1}{2} - V_2\frac{l_2}{2} + H_1y - H_2y\right)\left(-\frac{l_1}{2}\right)\frac{ds}{EI_c} \\
& + \int_0^{h_b}(-V_0 + V_1)\frac{ds}{EA_b} + \epsilon th_b + \int_0^{h_c}(-V_1 + V_2)(-1)\frac{ds}{EA_c} - \epsilon th_c = 0
\end{aligned}$$

Which becomes, when E is constant and summations are used,

$$\begin{aligned}
0 = {} & M_0\left(-\frac{l_1h_b}{2I_b}\right) + H_0\left(-\frac{R_0l_1h_b}{2I_b} - \frac{l_1h_b^2}{4I_b}\right) + V_0\left(\frac{l_0l_1h_b}{4I_b} - \frac{h_b}{A_b}\right) \\
& + M_1\left(\frac{l_1h_b}{2I_b} - \frac{l_1h_c}{2I_c}\right) + H_1\left(\frac{R_1l_1h_b}{2I_b} + \frac{l_1h_b^2}{4I_b} - \frac{l_1R_1h_c}{2I_c} - \frac{l_1h_c^2}{4I_c}\right) \\
& + V_1\left(2\sum_0^{\frac{S_1}{2}}\frac{x^2\Delta s_1}{I_1} + \frac{l_1^2h_b}{4I_b} + \frac{l_1^2h_c}{4I_c} + \frac{h_b}{A_b} + \frac{h_c}{A_c}\right) + M_2\left(\frac{l_1h_c}{2I_c}\right) \\
& + H_2\left(\frac{R_2l_1h_c}{2I_c} + \frac{l_1h_c^2}{4I_c}\right) + V_2\left(\frac{l_1l_2h_c}{4I_c} - \frac{h_c}{A_c}\right) + \epsilon t(h_b - h_c)E \quad (30)
\end{aligned}$$

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